ECE 4300 – Lasers and Optoelectronics Name: Athith Krishna
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Homework 1

Problem 1.1 (A qualitative overview of a laser.)
Describe succinctly (in ≤ 1 page!) in your own words & illustrations
a) how is laser light different from light from an incandescent bulb, and
b) what ingredients must be present to make a laser work.

Solution: a) LASER light is spectrally pure, → single wavelength, and they are coherent, i.e. all the photons are in phase. As a result, the beam of a laser light tends to stay as beam, and not diverge due to scattering. It works on the principle of stimulated transitions between the electron energy levels and is also highly amplified.
Incandescent bulb works by black body radiation. A tiny filament is heated up to extremely high temperatures and as heated elements radiate light, the bulb glows. The light emitted in this process has a whole range of energies (and wavelengths) and has no preferential direction of travel or polarization or phase. This light is not amplified. The differences can be summarized as follows:

Property	LASER	Incandescent Bulb
Nature of emission	Stimulated emission	Spontaneous Emission
Coherence	Coherent	Incoherent
Directionality	Highly directional (focused to a very small point)	Divergent (cannot be focused to a small point)
	Monochromatic	Polychromatic
	Amplified	Not amplified

b) The following ingredients must be present to make a laser to work:
(i) Lasing / Active / Gain medium
(ii) Optical Cavity
(iii) Resonator
(iv) External energy source (Pump)

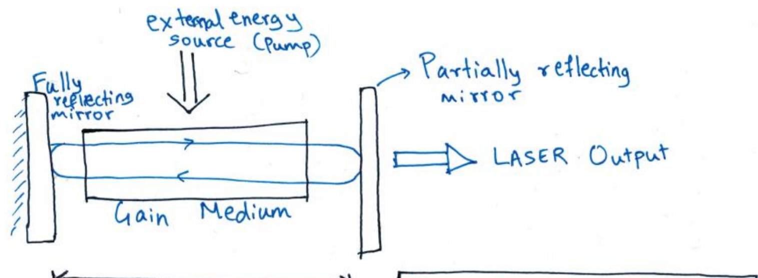


Figure 1 :- Basic components of a LASER

→Lasing medium is excited by the external energy source (pump) to produce population inversion. Spontaneous and stimulated emission of photons takes place, leading to the phenomenon of optical gain (amplification). Common medium include Ruby, He-Ne, YAG, etc.
→Pump provides energy required for the population inversion and stimulated emission to the entire system. Either electrical discharge or optical discharge can be used as pumping sources.
→Resonator guides the light about the simulated emission process induced by high-speed photons. There is also a fully reflective and a partially reflective mirror. Both are set up on optical axis, parallel to each other. The gain medium is located in the optical cavity between the two mirrors. This setup makes sure that only those photons which came along the axis, pass and others are reflected by the mirrors back into the medium, where it may be amplified by stimulated emission.

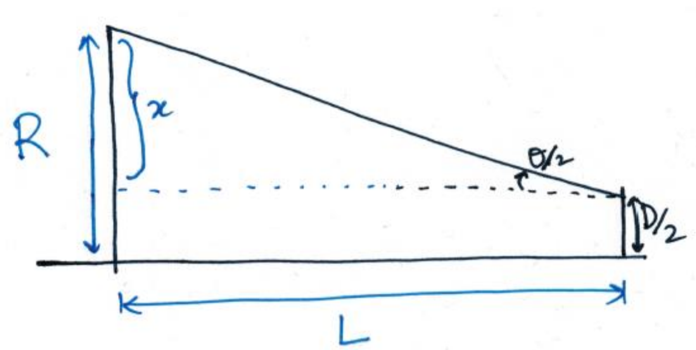
Problem 1.2 (Real-life problem you can solve as a Lasers expert!)

Verdeyen Problem # 1.4. Make a sketch to illustrate the problem.

Suppose that we are using an optical beam of diameter D to monitor the particle content of a column of gas. For many applications we would prefer to sample as small a volume as possible, and consequently we would first choose a very small beam. But if the path length is long, a very small beam would diverge quickly and thus sample a larger cross-sectional area of the gas column. Use the uncertainty relations to derive an expression for the beam diameter to minimize the volume of gas sampled. Assume a helium/neon probing laser ($\lambda = 632.8$ nm) and a simple cone describing the convergence and divergence of the beam envelope so as to evaluate for a gas column 10 m long.

Solution:

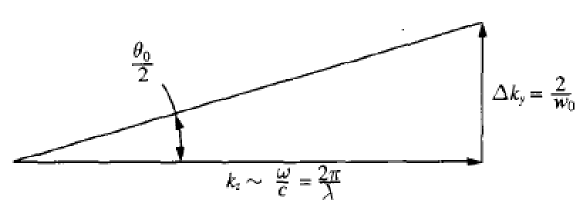
Sketch →



From the sketch, D is the diameter of the optical beam; L and R is the length and the radius of the resultant cone, respectively. $\frac{\theta}{2}$ is the angle of divergence of the beam.

$$R = \frac{D}{2} + x = \frac{D}{2} + L\tan\frac{\theta}{2} \approx \frac{D}{2} + \frac{L\theta}{2} \quad --(1) \text{ (small angles} \therefore \tan x = x)$$

From Uncertainty relationships and spread of the Gaussian beam, we know that:



$$\frac{\theta}{2} = \frac{\Delta k_y}{k_z} = \frac{\lambda}{\pi\left(\frac{D}{2}\right)} = \frac{2\lambda}{\pi D} --- (2)$$

From (2), (1) becomes,

$$R = \frac{D}{2} + \frac{2L\lambda}{\pi D}$$

Volume of the resultant cone becomes,

$$V = \frac{1}{3}\pi R^2 L = \frac{1}{3}\pi L\left[\left(\frac{D}{2}\right)^2 + \left(\frac{RD}{2}\right)^2 + R^2\right] --- (3)$$

Function to be minimized is $V\left(\frac{D}{L}\right)$. Putting value of R in (3) and differentiating gives,

$$\left[\frac{D}{L}\right]^4 = \frac{4}{3}\left[\frac{2\lambda}{\pi L}\right]^2 \Rightarrow D = \sqrt[4]{\frac{4}{3}\left\{\frac{2\times 10m \times 633nm}{\pi}\right\}^2} = 2.16\times 10^{-3}m$$

$$\frac{\theta}{2} = \frac{2\lambda}{\pi D} = 1.8\times 10^{-4}\text{rad}$$

Resultant cone diameter = $2R = 2\left(\frac{D}{2}+\frac{L\theta}{2}\right) = 5.89\times 10^{-3}m$

Answer: Minimum beam diameter → Expression is $D = \sqrt[4]{\frac{4}{3}\left\{\frac{2\times L\times\lambda}{\pi}\right\}^2}$ and the value for the given values is $D = 2.16\times 10^{-3}m$

Problem 1.3 (The Uncertainty Relations and Beam Spreading.)
Verdeyen Problem # 1.8.

The $TEM_{0,0}$ Gaussian beam has the smallest value of the product $\Delta x \Delta k_x = 1/2$ allowed by the uncertainty relationship. (The meaning of the terminology $TEM_{0,0}$ will be covered in Chapter 3.) The quantities Δx and Δk_x are to be interpreted as

$$\Delta x^2 = \int x^2|E(x)|^2dx \Big/ \int |E(x)|^2dx$$

$$\Delta k_x^2 = \int k_x^2|E(k_x)|^2dk_x \Big/ \int |E(k_x)|^2dx$$

with $E(x)$ and $E(k)_x)$ being related by the Fourier transform.

(a) What are the values for Δx and Δk_x for $E(x) = E_0\exp[-(x/w_0)^2]$ (i.e., $TEM_{0,0}$)?

(b) What is the uncertainty product for a field given by

$$E_{10} = (\sqrt{2}x/w)\exp[-(x^2+y^2)/w_0^2]$$

(c) Sketch the intensity $E_{10}\cdot E_{10}^*/2\eta_0$ as a function of x.

Solution:

(a) From the equations above,

$$\Delta x^2 = \frac{\int x^2|E(x)|^2\,dx}{\int|E(x)|^2\,dx} \quad ---(1)$$

$$\Delta k_x^2 = \frac{\int k_x^2|E(k_x)|^2\,dk_x}{\int|E(k_x)|^2\,dk_x} ---(2)$$

For $TEM_{0,0}$ case,

$$E(x) = E_0\exp\left[-\left(\frac{x}{w_0}\right)^2\right] ---(4)$$

Putting this is in equation (1), and integrating by method of substitution, we get,

$$\Delta x^2 = \frac{\int x^2 \left(E_0 \exp\left[-\left(\frac{x}{w_0}\right)^2\right]\right)^2 dx}{\int \left(E_0 \exp\left[-\left(\frac{x}{w_0}\right)^2\right]\right)^2 dx} \quad ---(3)$$

Let $t = \frac{\sqrt{2}x}{w_0} \Rightarrow \frac{dt}{dx} = \frac{\sqrt{2}}{w_0}$. Then (3) becomes,

$$\Delta x^2 = \frac{\int \frac{2x^2}{w_0^2} \exp\left[-\frac{2x^2}{w_0^2}\right] d\left(\frac{\sqrt{2}x}{w_0}\right) w_0^3/(2^{\frac{3}{2}})}{\int \exp\left[-\frac{2x^2}{w_0^2}\right] d\left(\frac{\sqrt{2}x}{w_0}\right) w_0/\sqrt{2}} = \frac{\frac{w_0^2}{2}\left(\int_{-\infty}^{\infty} t^2 \exp(-t^2)\, dt\right)}{\int_{-\infty}^{\infty} \exp(-t^2)\, dt}$$

Evaluating the above expression on Wolfram Alpha, we get,

$$\Delta x^2 = \frac{w_0^2}{2}\left(\frac{1}{2}\right) \Rightarrow \Delta x = \frac{w_0}{2}$$

Expression for $E(x)$ - (4) above in k-space can be written as,

$$E(k_x) = E_0 \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x}{w_0^2}\right)^2\right] \exp(-jk_x x)\, dx = \sqrt{\pi} w_0 E_0 \exp\left(-\left(\frac{k_x w_0}{2}\right)^2\right)$$

Putting this expression in Eq (2), we get,

$$\Delta k_x^2 = \frac{\int k_x^2 \left[\sqrt{\pi w_0 E_0 \exp\left(-\left(\frac{k_x w_0}{2}\right)^2\right)}\right]^2 dk_x}{\int \left[\sqrt{\pi w_0 E_0 \exp\left(-\left(\frac{k_x w_0}{2}\right)^2\right)}\right]^2 dk_x} \quad ----(5)$$

Let $k_t = \frac{w_0 k_x}{\sqrt{2}} \Rightarrow \frac{d(k_t)}{dk_x} = \frac{w_0}{\sqrt{2}}$. Then (5) becomes,

$$\Delta k_x^2 = \frac{\left\{\frac{w_0 k_x}{2}\right\}^2 \int \exp - \left[\frac{w_0^2 k_x^2}{2}\right] d\left(\frac{w_0 k_x}{\sqrt{2}}\right)(2^{\frac{3}{2}})/w_0^3}{\int \exp - \left[\frac{w_0^2 k_x^2}{2}\right] d\left(\frac{w_0 k_x}{\sqrt{2}}\right)\sqrt{2}/w_0} = \frac{2}{w_0^2}\left\{\frac{\left(\int_{-\infty}^{\infty} k_t^2 \exp(-k_t^2)\, dk_t\right)}{\int_{-\infty}^{\infty} \exp(-k_t^2)\, dk_t}\right\}$$

Evaluating the above expression on Wolfram Alpha, we get,

$$\Delta k_x^2 = \frac{2}{w_0^2}\left(\frac{1}{2}\right) \Rightarrow \boxed{\Delta k_x = \frac{1}{w_0^2}}$$

(b) The given field is,

$$E_{10} = \left(\frac{\sqrt{2}x}{w_0}\right) \exp\left[-\frac{x^2 + y^2}{w_0^2}\right] ----(6)$$

This can be written in k-space as,

$$E_{10}(k_x) = E_0 \int_{-\infty}^{\infty} \frac{\sqrt{2}x}{w_0} \exp\left(-\left[\frac{x}{w_0}\right]^2\right) \exp(-jk_x x)\, dx$$

Let $t = \frac{x}{w_0} + j\frac{k_x w_0}{z} \rightarrow \frac{\sqrt{2x}}{w_0} = \sqrt{2}t - j\frac{k_x w_0}{\sqrt{2}}$

$$E_{10}(k_x) = w_0 exp - \left[\frac{k_x w_0}{2}\right]^2 \left(\int_{-\infty}^{\infty} \exp(-t^2)\sqrt{2}\, t\, dt - j\frac{k_x w_0}{\sqrt{2}} \int_{-\infty}^{\infty} \exp(-t^2)\ dt\right)$$

$$= j\frac{k_x w_0}{\sqrt{2}}\, e^{-\left(\frac{k_x w_0}{2}\right)^2} \sqrt{\pi} - - - - (7)$$

Putting (6) in (1), and integrating by substitution, we get,

$$\Delta x^2 = \frac{\frac{w_0^2}{2}\int_{-\infty}^{\infty} t^4 \exp(-t^2)\, dt}{\int_{-\infty}^{\infty} t^2 \exp(-t^2)\, dt} = \frac{w_0^2}{2}\left(\frac{3}{2}\right) \Rightarrow \Delta x = \frac{\sqrt{3}w_0}{2}$$

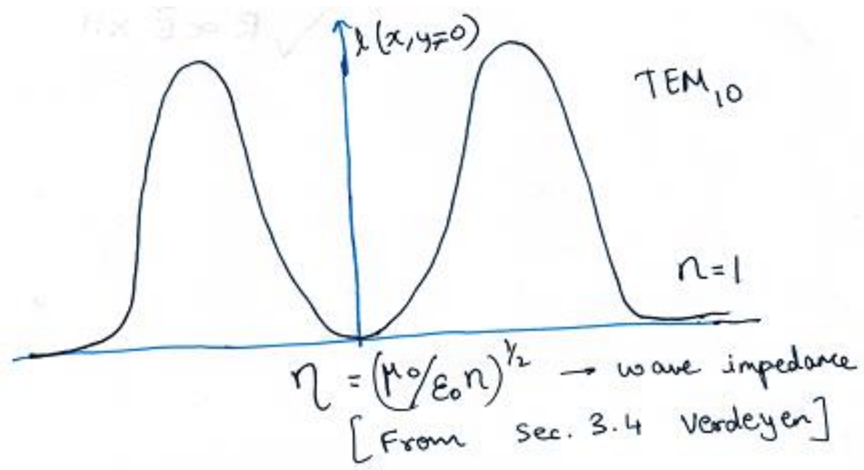
Putting (7) in (2), and integrating by substitution, we get,

$$\Delta k_x^2 = \frac{\frac{2}{w_0^2}\int_{-\infty}^{\infty} k_t^4 \exp(-k_t^2)\, dk_t}{\int_{-\infty}^{\infty} k_t^2 \exp(-k_t^2)\, dk_t} = \frac{2}{w_0^2}\left(\frac{3}{2}\right) \Rightarrow \Delta k_x = \frac{\sqrt{3}}{w_0}$$

Uncertainty product is given by,

$$\boxed{\Delta x \Delta k_x = \frac{\sqrt{3}w_0}{2}\frac{\sqrt{3}}{w_0} = \frac{3}{2}}$$

(c) Sketch of intensity vs x:



Problem 1.4 (Brush up your EMag skills.)
Verdeyen Problem # 1.3.

The algebraic forms for Maxwell's equations for a linear homogeneous anisotropic medium are

$$\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D}$$

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$$

where **B** is related to **H** and **D** to **E** by

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

For many materials, the polarization vector **P** is not collinear with **E**; hence, **D** is not collinear with **E** either. The same comments apply to **B**, **M**, and **H**. Assume a dielectric medium with $\mathbf{M} = 0$ but with no restrictions placed on **D** and **E**.

(a) Show that $\mathbf{k} \cdot \mathbf{D} = 0$.
(b) Show that the wave vector **k** always points in the direction of $\mathbf{D} \times \mathbf{B}$.
(c) Show that the amplitude of the wave vector **k** is given by

$$\mathbf{k}^2 = \omega^2 \mu_0 \frac{\mathbf{D} \cdot \mathbf{D}}{\mathbf{E} \cdot \mathbf{D}}$$

(d) Show that the Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}^*/2$, can point in a direction other than that of the wave vector **k**.

Solution:

(a) We are given that $\boldsymbol{k} \times \boldsymbol{H} = -\omega \boldsymbol{D} --- (1)$
Taking a dot product of that with $\boldsymbol{k}$**,** we have,

$$\boldsymbol{k} \cdot (\boldsymbol{k} \times \boldsymbol{H}) = 0 --- (2)$$

This is because cross product of 2 vectors is normal to both the vectors. Now, putting (2) in (1), we get,

$$\boldsymbol{k} \cdot (-\omega \boldsymbol{D}) = 0 \rightarrow \boxed{\boldsymbol{k} \cdot \boldsymbol{D} = \boldsymbol{0}} \text{ as required. ---(3)}$$

This is because, $\omega \neq 0$.

(b) We are given that $\boldsymbol{k} \times \boldsymbol{H} = -\omega \boldsymbol{D} \Rightarrow \boldsymbol{D} = -\left(\frac{1}{\omega}\right)(\boldsymbol{k} \times \boldsymbol{H}) --- (4)$

$$\boldsymbol{k} \times \boldsymbol{E} = \omega \boldsymbol{B} \Rightarrow \boldsymbol{B} = \left(\frac{1}{\omega}\right)(\boldsymbol{k} \times \boldsymbol{E}) --- (5)$$

Using the following property of cross products,

$$(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = \boldsymbol{C}[\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{D})] - \boldsymbol{D}[\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})]$$

From (4) & (5), we have

$$\boldsymbol{D} \times \boldsymbol{B} = -\left(\frac{1}{\omega^2}\right)\{\boldsymbol{k}[\boldsymbol{k} \cdot (\boldsymbol{H} \times \boldsymbol{E})] - \boldsymbol{E}[\boldsymbol{k} \cdot (\boldsymbol{H} \times \boldsymbol{k})]\} --- (6)$$

From (2), the second term goes to 0 and $\boldsymbol{k} \cdot (\boldsymbol{H} \times \boldsymbol{E}) \neq \boldsymbol{0}$ and is a scalar Therefore, 6 becomes:

$$\boldsymbol{D}\times\boldsymbol{B} = \underbrace{-\left(\frac{1}{\omega^2}\right)[\boldsymbol{k}\cdot(\boldsymbol{H}\times\boldsymbol{E})]}_{scalar}(\boldsymbol{k}) - - - (7)$$

equation (7) shows that wave vector, $\boldsymbol{k}$ and $\boldsymbol{D}\times\boldsymbol{B}$ are in the same direction.

(c)

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\times(\omega\boldsymbol{B}) = \omega\big(\boldsymbol{k}\times(\mu_0\boldsymbol{H})\big) = \omega\mu_0(\boldsymbol{k}\times\boldsymbol{H}) = \omega\mu_0(-\omega\boldsymbol{D})$$
$$= -\omega^2\mu_0\boldsymbol{D} - - - (8)$$
$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) = [\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E}) - k^2\boldsymbol{E}] = -\omega^2\mu_0\boldsymbol{D} - -(9)$$

Taking dot product of both sides of (9) with $\boldsymbol{D}$, we get

$$\boldsymbol{D}\cdot[\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E}) - k^2\boldsymbol{E}] = -\omega^2\mu_0\boldsymbol{D}.\boldsymbol{D}$$
$$\underbrace{\boldsymbol{D}\cdot\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E})}_{\boldsymbol{D}.\boldsymbol{k}=\boldsymbol{0}\ (from\ a)} - \boldsymbol{D}\cdot k^2\boldsymbol{E} = -\omega^2\mu_0\boldsymbol{D}.\boldsymbol{D} - -(10)$$

$$\Rightarrow \boxed{k^2 = \omega^2\mu_0\frac{\boldsymbol{D}.\boldsymbol{D}}{\boldsymbol{D}.\boldsymbol{E}}}$$

(d) In anisotropic medium, Poynting vector, $\boldsymbol{S} = \left(\frac{1}{2}\right)\boldsymbol{E}\times\boldsymbol{H}^*$

$$\boldsymbol{k}\times\boldsymbol{E} = \omega\boldsymbol{B} = \omega\mu_0\boldsymbol{H}$$
$$\boldsymbol{S} = \left(\frac{1}{2}\right)\boldsymbol{E}\times\frac{-\boldsymbol{E}\times\boldsymbol{k}}{\omega\mu_0}$$

Let us take the case of a linear anisotropic medium.

$\vec{D} = \varepsilon_0[\varepsilon]\vec{E}$ $\boldsymbol{D}$ is no longer parallel to $\boldsymbol{E}$

$\vec{k}.\vec{D} = 0$, $\vec{k}.\vec{B} = 0$ -> **D** and **B** are still orthogonal to **k**

$\vec{k}\times\vec{H} = -\omega\vec{D}$ -> **D** and **H** are orthogonal

$\vec{k}\times\vec{E} = \omega\vec{B}$ -> **E** and **B** are orthogonal

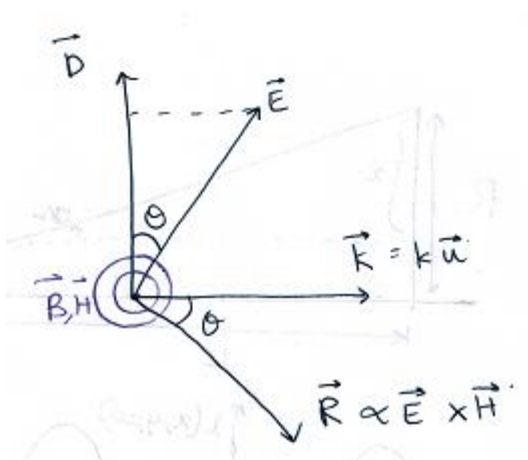
} **D, E and k are in the same plane, orthogonal to B**

From the attached sketch it can be seen that

$$\boldsymbol{D} = -\frac{1}{\mu_0\omega^2}\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) = \frac{k^2}{\mu_0\omega^2}[\boldsymbol{E} - (\boldsymbol{u}\cdot\boldsymbol{E})\boldsymbol{u}]$$

$\boldsymbol{D}$ is the projection of $\boldsymbol{E}$ in the plane orthogonal to $\boldsymbol{u}$ (to within a mult. factor)

Therefore the Poynting vector (direction of « light ray») is no longer parallel to the wave vector (direction of propagation of the phase).



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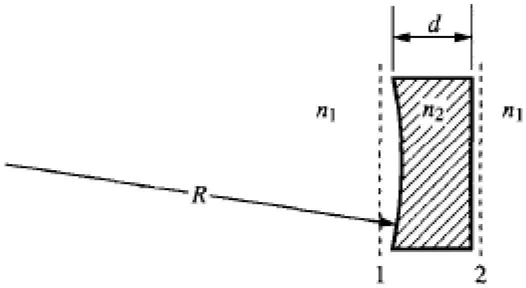
Problem 1.5 (on Ray Tracing.)

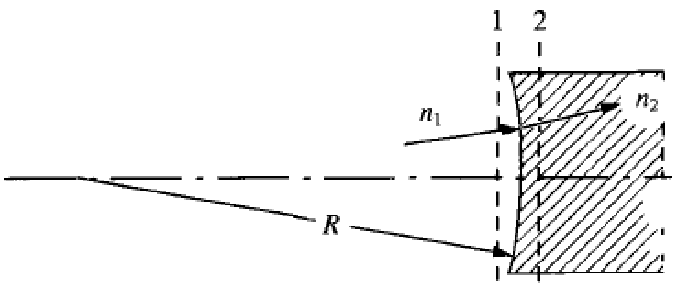
Verdeyen Problem # 2.4.

Combine the results of problems 2.1 and 2.2 to derive the ray matrix for the negative lens. (Assume that $R \gg d$.)



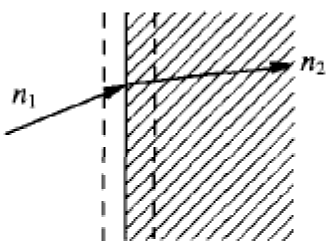
Solution:

Problem 2.1 gives us the following:



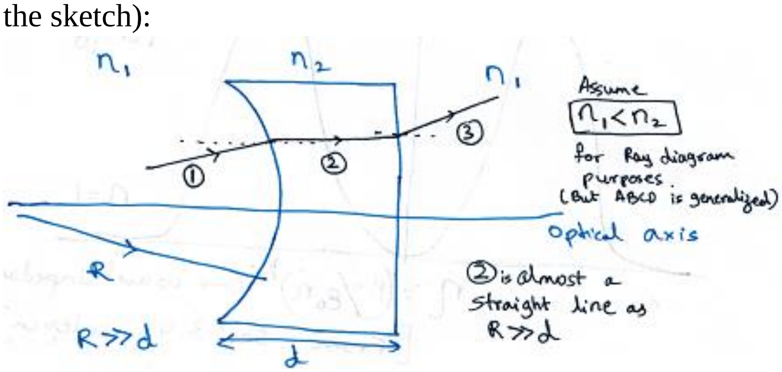
Ans.: $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ \left(1 - \frac{n_1}{n_2}\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$

Problem 2.2 gives us the following:



Ans.: $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$

Combining the two for the negative lens, we will have the following steps of propagation (from the sketch):



(i) Propagation from medium of n_1 refractive index at spherical dielectric interface to a medium of index n_2 (Result from problem 2.1) – Ray part 1

(ii) Propagation in a dielectric(same medium) of refractive index n_2 for a distance $'d'$ - Ray part 2

(iii) Propagation from medium of n_2 refractive index at plane dielectric interface to a medium of index n_1 ((Result from problem 2.2) – Ray part 3

Thus the ABCD matrix of the entire system can be written as:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix}}_{ray\ part\ 3} \cdot \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{ray\ part\ 2} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ \left(1-\frac{n_1}{n_2}\right)\left(\frac{1}{R}\right) & \frac{n_1}{n_2} \end{bmatrix}}_{ray\ part\ 1} = \begin{bmatrix} 1 & d \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \left(1-\frac{n_1}{n_2}\right)\left(\frac{1}{R}\right) & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\left(\frac{d}{R}\right)\left(1-\frac{n_1}{n_2}\right) & \frac{dn_1}{n_2} \\ \frac{n_2}{n_1 R}\left(1-\frac{n_1}{n_2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1+\left(\frac{d}{R}\right)\left(1-\frac{n_1}{n_2}\right) & \frac{dn_1}{n_2} \\ \frac{1}{R}\left(\frac{n_2}{n_1}-1\right) & 1 \end{bmatrix}$$

Verification: $\det(ABCD) = 1$

The resultant ABCD matrix for the entire system is as follows:

$$\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+\left(\frac{d}{R}\right)\left(1-\frac{n_1}{n_2}\right) & \frac{dn_1}{n_2} \\ \frac{1}{R}\left(\frac{n_2}{n_1}-1\right) & 1 \end{bmatrix}}$$

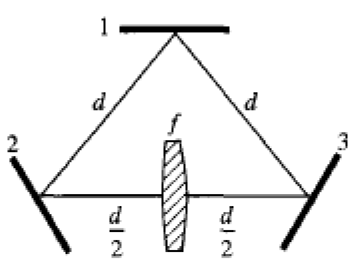
Problem 1.6 (Stability criteria for a Ring-Laser cavity.)
Verdeyen Problem # 2.7.

Consider the ring laser cavity shown in the accompanying diagram.

(a) Show an equivalent-lens waveguide for this cavity and identify a unit cell starting just after the lens and proceeding counterclockwise around the triangle.

(b) What is the transmission matrix for this unit cell? (Demonstrate that you have the component matrices in proper order.)

(c) What are the values of d/f that make this a stable cavity?

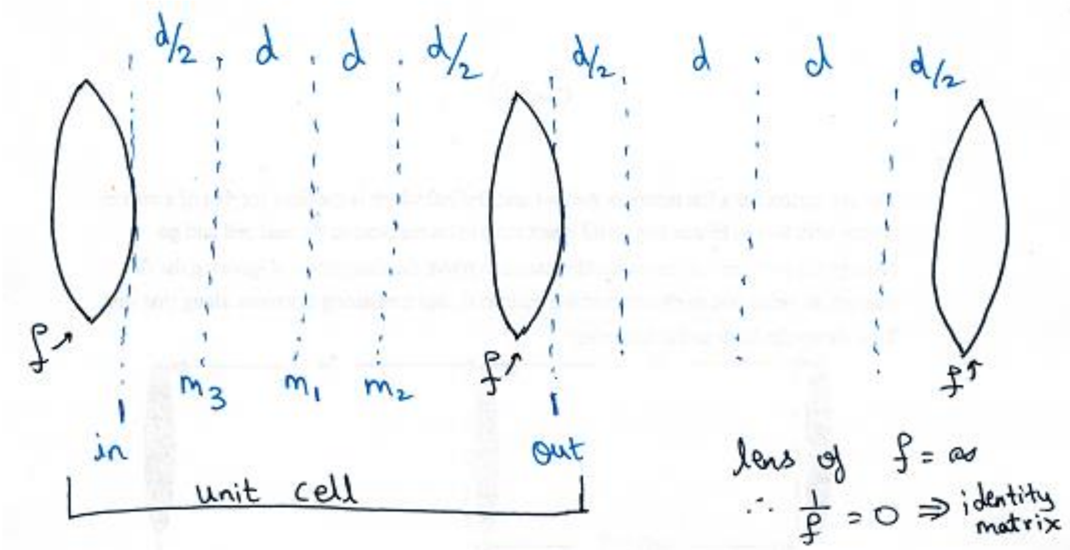


Solution:
(a) For a plane mirror, the ABCD matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can see that this is similar to that of a spherical mirror with $R \to \infty$. The plane mirrors can be considered just as directors of the optical axis and can be ignored in a unit cell for propagation as their ray matrix is just an identity matrix.

The resultant equivalent lens arrangement is:



(b) To obtain the ABCD matrix for the system,

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 3d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3d \\ -\frac{1}{f} & 1 - \left(\frac{3d}{f}\right) \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

$$\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 3d \\ -\frac{1}{f} & 1 - \left(\frac{3d}{f}\right) \end{bmatrix}}$$

(c) Values of $\frac{d}{f}$ that make the cavity stable: Stable if -

$$-1 < \frac{A+D}{2} < 1$$

$$-1 < \frac{2 - \frac{3d}{f}}{2} < 1 \Rightarrow -2 < 2 - \frac{3d}{f} < 2 \Rightarrow -4 < -\frac{3d}{f} < 0 \Rightarrow -\frac{4}{3} < -\frac{d}{f} < 0 \Rightarrow 0 < \frac{d}{f} < \frac{4}{3}$$

Values of $\frac{d}{f}$ that make the cavity stable are $\rightarrow \boxed{\frac{d}{f} \in (0, \frac{4}{3})}$