ECE 4300: Lasers & Optoelectronics
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Assignment 2

Problem 2.1 (Gain and escape from an unstable cavity)

Verdeyen Problem # 2.10.

2.10. Consider the obviously unstable cavity shown in the accompanying diagram. Suppose that a ray starts out at the flat mirror with a zero slope and at a position that is one tenth of the radius of the mirror.

(a) Show the equivalent-lens waveguide for the cavity. Identify two unit cells, one starting at the flat mirror and the second starting just before the spherical mirror.

(b) Since the cavity is unstable, we should return to the second-order difference equation to obtain a solution of the form (use unit cell 1)

$$r_s = r_a(F_+)^s + r_b(F_-)^s$$

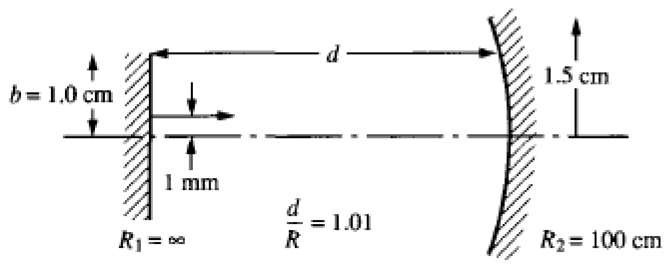
(1) What are the values of F_+ and F_-?

(2) What are the values of r_a and r_b?

(c) How many passes through the cavity does the ray make before it misses the flat mirror?

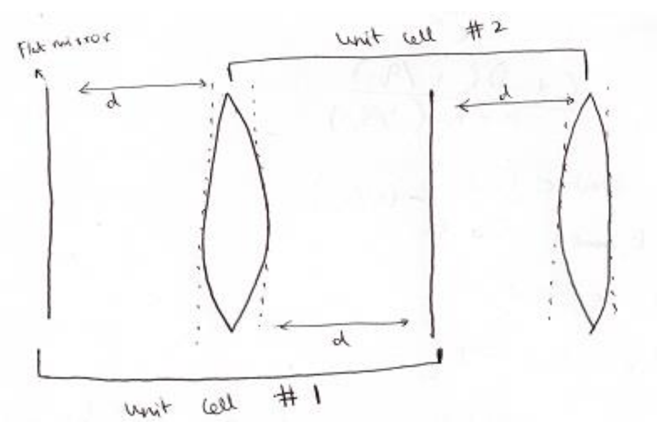
(d) Where does the ray leave the cavity: at the flat mirror or at the curved one? How many passes does it make? (To answer this, you will have to repeat (b) and (c) for the second unit cell and then decide.)

(e) If the beam associated with this ray started with 1 μW of power and the power gain per pass $G = 5$, what is the power leaving the cavity? (Assume that the mirrors are perfectly reflecting.)



Solution:

(a) The equivalent lens waveguide for the given setup is:



Both the unit cells – (1) Starting at the flat mirror and (2) Starting before the spherical mirror, are marked in the sketch

(b) We first need to assemble the $ABCD$ matrix for unit cells.
For the first unit cell: starting from the flat mirror,

$$ABCD_1 = \begin{bmatrix} 1-\frac{d}{f} & d\left(2-\frac{d}{f}\right) \\ -\frac{1}{f} & 1-\frac{d}{f} \end{bmatrix} --- (1)$$

For the first unit cell, using the second order difference equation, 2.10.2,

$$rF^s\left[F^2 - 2\left(\frac{A+D}{2}\right)F + 1\right] = 0$$

$$F_1 \text{ (here } F_+) = \frac{A+D}{2} + \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} --- (2)$$

$$F_2 \text{ (here } F_-) = \frac{A+D}{2} - \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} --- (3)$$

From (1), we know that, $\frac{A+D}{2} = \frac{\left(1-\frac{d}{f}\right)+\left(1-\frac{d}{f}\right)}{2} = 1 - \frac{d}{f}$

We also know from the given diagram that $\frac{d}{R} = 1.01 \rightarrow \frac{A+D}{2} = 1 - \frac{d}{f} = 1 - 2.02 = -1.02$

Putting this value in (2) and (3), we get (b) part 1,

$$F_+ = \frac{A+D}{2} + \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} = -1.02 + [(-1.02)^2 - 1]^{\frac{1}{2}} = \boxed{-0.819} ----(4)$$

$$F_- = \frac{A+D}{2} - \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} = -1.02 - [(-1.02)^2 - 1]^{\frac{1}{2}} = \boxed{-1.221} ----(5)$$

(b) Part 2

Using the given solution for unstable cavity, $r_s = r_a(F_+)^s + r_b(F_-)^s$

We have the following equations 2.10.7 and 2.10.8,

$$r_b = \frac{1}{F_+ - F_-}[a(F_+ - A) - Bm]$$

$$r_a = \frac{1}{F_- - F_+}[a(F_- - A) - Bm]$$

where $a = r_a + r_b$,

From (4) and (5), we get values of r_a and r_b,

$$r_a = r_b = \boxed{0.005}$$

(c) Putting the values of F_+, F_-, r_a, r_b in to the solution equation, we get,

$$r_s = 0.005(-0.819)^s + 0.005(-1.221)^s --- (6)$$

We check at what value of s, Eq (6) becomes greater than 1, solving in MATLAB, we get, $r_s > 1$ when $s \geq 15$.

Therefore the ray misses the flat mirror after $\boxed{s = 15 \text{ round trips.}}$

(d) We first need to assemble the $ABCD$ matrix for unit cell #2.
For the second unit cell: starting before the spherical mirror,

$$ABCD_1 = \begin{bmatrix} 1 - \dfrac{2d}{f} & 2d \\ -\dfrac{1}{f} & 1 \end{bmatrix} --- (7)$$

For the second unit cell, using the second order difference equation, 2.10.2,

$$rF^s \left[F^2 - 2\left(\frac{A+D}{2}\right)F + 1\right] = 0$$

Using equations (2) and (3) for F_+ and F_-,

From (7), we know that, $\frac{A+D}{2} = \frac{\left(1-\frac{2d}{f}\right)+(1)}{2} = 1 - \frac{d}{f}$
We also know from the given diagram that $\frac{d}{R} = 1.01 \rightarrow \frac{A+D}{2} = 1 - \frac{d}{f} = 1 - 2.02 = -1.02$

Putting this value in (2) and (3),

$$F_+ = \frac{A+D}{2} + \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} = -1.02 + [(-1.02)^2 - 1]^{\frac{1}{2}} = -0.819 ----(8)$$

$$F_- = \frac{A+D}{2} - \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{\frac{1}{2}} = -1.02 - [(-1.02)^2 - 1]^{\frac{1}{2}} = -1.221 ----(9)$$

Using the given solution for unstable cavity, $r_s = r_a(F_+)^s + r_b(F_-)^s$

We have the following equations 2.10.7 and 2.10.8,

$$r_b = \frac{1}{F_+ - F_-}[a(F_+ - A) - Bm]$$

$$r_a = \frac{1}{F_- - F_+}[a(F_- - A) - Bm]$$

where $a = r_a + r_b$,

From (8) and (9), we get values of r_a and r_b,

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$$r_a = -0.4525; r_b = 0.5525$$

Putting the values of F_+, F_-, r_a, r_b in to the solution equation, we get,

$$r_s = -0.4525(-0.819)^s + 0.5525(-1.221)^s --- (10)$$

We check at what value of s, Eq (10) becomes greater than 1, solving in MATLAB, we get, $r_s > 1$ when $s = 6$. Putting this value in (10) we get the value of r where the ray leaves, $\boxed{r = 1.69\ cm.}$

The ray will leave the cavity at the spherical mirror after $2s$ round trips + 1 pass = $\boxed{\text{12 roundtrips+ 1 pass}}$

(e) We know that the ray started with $1\mu W$ power. Each pass (13 in all) will lead to an amount of gain, $G = 5$. Thus, the output power is:

$$P_o = P_i \times G^{13} = 1\mu W \times (5)^{13} = \boxed{1.22\ kW}$$

Problem 2.2 (Basics of Gaussian Beams and Laser Fields & Photons)
Verdeyen Problem # 3.2.

3.2. **(a)** A certain commercial helium/neon laser is advertised to have a farfield divergence angle of 1 milliradian at $\lambda_0 = 632.8$ nm. What is the spot size w_0?
(b) The power emitted by this laser is 5 mW. What is the peak electric field in volts per centimeter at $r = z = 0$?
(c) How many photons per second are emitted by this laser beam?
(d) Electromagnetic energy can only come in packages of $h\nu$. If one more photon per second were emitted by this laser, what is the new power specification? (The point of this part of the problem is to recognize that there is a time and a place for making the distinction between a classical field and a photon: Should we start here?)

Solution:

(a) From equation 3.4.2 we have,

$$\frac{\theta}{2} = \frac{dw}{dz} = \frac{\lambda_0}{\pi n w_0} \Rightarrow w_0 = \frac{2\lambda_0}{\pi\theta} = \frac{2(632.8\text{nm})}{\pi(10^{-3}\text{rad})} = \boxed{4.03 \times 10^{-2}\text{cm}}$$

(b) From equation 3.4.3 we have,

$$\text{P} = \frac{\frac{1}{2}E_0^2}{\eta_0}\left(\frac{\pi w_0^2}{2}\right) \text{ where } \eta_0 = \left(\frac{\mu_0}{\epsilon_0 n}\right)^{\frac{1}{2}} - \text{wave impedance}$$

$$E_0 = \sqrt{\frac{4\text{P}\eta_0}{\pi w_0^2}} = \sqrt{\frac{4\times 5\text{mW}\times\eta_0}{\pi(4.03\times 10^{-2}\text{cm})^2}} = \boxed{38.5\ \text{V/cm}}$$

(c) Number of photons/second, $\mathcal{N}$

$$\mathcal{N} = \frac{\mathrm{P}}{Energy} = \frac{\mathrm{P}}{h\nu} = \frac{\mathrm{P}}{\frac{hc}{\lambda}} = \frac{5\mathrm{mW}}{hc/(632.8\mathrm{nm})} = \boxed{1.59 \times 10^{16}\mathrm{photons}/s}$$

(d) P=$\mathcal{N}h\nu$. If there was one more photon,

$$\frac{dP}{d\mathcal{N}} = h\nu = \frac{hc}{\lambda} \Rightarrow \Delta P = 3.14 \times 10^{-19}\mathrm{W}$$

This new ΔP is very small compared to the previous power specification of $P = 5\mathrm{mW}$ therefore increase in one photon will not lead to a considerable change to the new power specification.

Problem 2.3 (More practice in Gaussian Beam Propagation)
Verdeyen Problem # 3.4.

3.4. A 10-W argon ion laser oscillating at 4880 Å has a minimum spot size of 2 mm.
- **(a)** How far will this beam travel before the spot size is 4 mm?
- **(b)** What fraction of the 10 W is contained in a hole of *diameter* $2w(z)$?
- **(c)** Express the frequency/wavelength of this laser in eV, nm, μm, ν(Hz), and $\bar{\nu}(\mathrm{cm}^{-1})$.
- **(d)** What is the amplitude of the electric field when $w = 1$ cm?

Solution:

(a) We have the following data: $w_0 = 2\mathrm{mm}$;
From equation 3.3.7, we have

$$z_0 = \frac{\pi n w_0^2}{\lambda_0} = \frac{\pi(1)(2\mathrm{mm})^2}{488\mathrm{nm}} = 25.75\mathrm{m}$$

Now, using equation 3.3.10, we have

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right] \Rightarrow z = z_0\left(\sqrt{\frac{w^2(z)}{w_0^2} - 1}\right) = 25.75\mathrm{m}\sqrt{\frac{(4\mathrm{mm})^2}{2\mathrm{mm}^2} - 1} = \sqrt{3}(25.75\mathrm{m})$$

$$\Rightarrow \boxed{z = 44.6\ \mathrm{m}}$$

(b) Diameter of the hole, $D = 2w(z)$; $0 \le w \le r$

$$\frac{P(r < w)}{P_{total}} = \frac{\int_0^{2\pi}\int_0^{r=w} I(r) \cdot r\,dr\,d\phi}{\int_0^{2\pi}\int_0^{\infty} I(r) \cdot r\,dr\,d\phi} = \frac{\int_0^{2\pi}\int_0^{w} I_0 \exp\left(-\frac{2r^2}{w^2}\right) \cdot r\,dr\,d\phi}{\int_0^{2\pi}\int_0^{\infty} I_0 \exp\left(-\frac{2r^2}{w^2}\right) \cdot r\,dr\,d\phi} --- (1)$$

Substitute $x = \frac{2r^2}{w^2}$; $dx = \frac{4r}{w^2}dr$ in (1)

$$\frac{P(r<w)}{P_{total}} = \frac{2\pi I_0\left(\frac{w^2}{4}\right)\int_0^2 \exp(-x)\,dx}{2\pi I_0\left(\frac{w^2}{4}\right)\int_0^\infty \exp(-x)\,dx} = \frac{\int_0^2 \exp(-x)\,dx}{\int_0^\infty \exp(-x)\,dx} = (e^{-2}-1)/(e^{\infty}-1)$$
$$= 1-\exp(-2) = \boxed{0.865}$$

(c)

$$\lambda_0 = 4880\dot{A} \rightarrow \nu = \frac{c}{\lambda_0} = 6.15\times 10^{14} Hz$$

eV	$\frac{h\nu}{q_0} = h\cdot\frac{6.15\times 10^{14}Hz}{1.6\times 10^{-19}C} = 2.54\ eV$
nm	$\lambda_0 = 4880\dot{A} = 488nm$
μm	$\lambda_0 = 4880\dot{A} = 0.488\mu m$
$\nu(Hz)$	$\nu = \frac{c}{\lambda_0} = 6.15\times 10^{14}Hz$
$\bar{\nu}(cm^{-1})$	$\bar{\nu}(cm^{-1}) = \frac{1}{\lambda_0} = 20491.8cm^{-1}$

(d)

We know that $\frac{E(z)}{E(z_0)} = \frac{w_0}{w(z)}$

Using equation 3.4.3,

$$\mathrm{P} = \frac{\frac{1}{2}E_0^2}{\eta_0}\left(\frac{\pi w_0^2}{2}\right) \Rightarrow 10W = \frac{\frac{1}{2}E_0^2}{\eta_0}\left(\frac{\pi w_0^2}{2}\right) \Rightarrow E(z_0) = 346V/cm$$

$$\frac{E(z)}{346V/cm} = \frac{2mm}{1cm} \Rightarrow \boxed{E(z) = 69.3\ V/cm}$$

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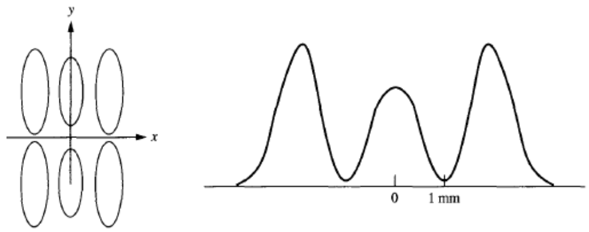
Problem 2.4 (Higher order Laser modes)

Verdeyen Problem # 3.7.

3.7. The intensity of a laser has the following visual appearance when projected on a surface.

(a) Name the mode (i.e., $\text{TEM}_{m,p}$; $m = ?$; $p = ?$).

(b) A plot of the relative *intensity of another* mode as a function of x (for $y = 0$) is shown below at the right. The variation with respect to y is a simple bell-shape curve. What is the spot size w?



Solution:

(a) From the dot diagram of the mode we can see that the number of dots = $(m + 1)$ and $(p + 1) = (3)(2) \rightarrow m = 2, p = 1$

The given mode is $\boxed{\text{TEM}_{2,1} \text{ mode.}}$

(b) The given mode is the $\text{TEM}_{2,0}$ mode.

Using equation 3.5.2 to get the Hermite polynomial of order $m = 2$, we get $H_2(u) = 2u^2 - 1$

Setting this equal to 0, $2u^2 - 1 = 0 \Rightarrow u = \pm\frac{1}{\sqrt{2}}$

$$E = E_0 H_2(u)\left(\frac{w_0}{w(z)}\exp\left(-\frac{x^2+y^2}{w}\right)\right)\exp(-\mathrm{j}\phi_{\mathrm{e}})\exp(-\mathrm{j}\phi_{\mathrm{r}}) = 0$$

$$u = \frac{\sqrt{2}x}{w(z)} = \pm\frac{1}{\sqrt{2}} \Rightarrow x|_{E=0} = \frac{w(z)}{2} \Rightarrow \boxed{w(z) = 2\text{mm}}$$

Problem 2.5 (Gaussian Beam ABCD Matrix for a Nonuniform Mirror)
Verdeyen Problem # 3.19.

3.19. The $ABCD$ matrix for a flat mirror with uniform reflectivity is trivial with $A = D = 1$ and $B = C = 0$. This problem concerns a nonuniform flat mirror with a reflectivity that is "tapered" with radius

$$\Gamma^2(r) = \exp[-(tr)^2]$$

Assume that a $\mathrm{TEM}_{0,0}$ Gaussian beam impinges on such a mirror (with the axis of the beam corresponding to the axis of the mirror). Find a new $ABCD$ matrix for this tapered mirror such that the $ABCD$ law still applies for this element.

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

where $q_{1,2}$ is the complex beam parameter of the incident and the reflected wave, respectively. (NOTE: Your answer must reduce to the usual one when $t = 0$. Do not waste your time tracing rays.)

Solution:

We know from ABCD law that

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \to \frac{1}{q_2} = \frac{\left(C + D\left(\frac{1}{q_1}\right)\right)}{A + B\left(\frac{1}{q_1}\right)} \quad --- (1)$$

We are given the reflectivity of the mirror as, $\Gamma^2(r) = \exp[-(tr)^2]$. In normal uniform mirror, t=0, but here t≠0. Getting the reflection coefficient from the reflectivity, we have

$\exp\left(-\left(\frac{tr)^2}{2}\right)\right) = \Gamma$

From the wave equation, $\nabla_t^2 E + \frac{\partial E}{\partial z^2} + \frac{\omega^2}{c^2} E = 0$, we can figure out the incident and the reflected field,

$$E_i = E_0 \exp\left(-\left(\frac{r^2}{w^2}\right)\right)$$

$$E_r = E_0 \exp\left(\frac{t^2}{2} - \frac{1}{w^2}\right) r^2$$

The complex beam parameter can be expressed as its reciprocal in the following form,

$$\frac{1}{q_2} = \frac{1}{R_1} - \frac{j\lambda}{\pi}\left(\frac{1}{w_1^2} + \frac{t^2}{2}\right) --- (2)$$

Eq (2) works because the mirror surface is flat which leads to all wave front curvature to be consistent.

We also know that $\frac{1}{q_1} = \frac{1}{R_1} - \frac{j\lambda}{\pi w_1^2}$ (Verdeyen P.78).

Thus (2) becomes,

$$\frac{1}{q_2} = \frac{1}{R_1} - \frac{j\lambda}{\pi}\left(\frac{1}{w_1^2} + \frac{t^2}{2}\right) = \frac{1}{R_1} - \frac{j\lambda}{\pi w_1^2} - \frac{\frac{j\lambda}{\pi}t^2}{2} = \frac{1}{q_1} - \frac{\frac{j\lambda}{\pi}t^2}{2} \quad --- (3)$$

From Eq. (1) we have,

$$\frac{1}{q_2} = \frac{Cq_1 + D}{Aq_1 + B} = \frac{\left(C + D\left(\frac{1}{q_1}\right)\right)}{A + B\left(\frac{1}{q_1}\right)}$$

Comparing this with Eq(3), we get the elements of the ABCD matrix for the given setup.

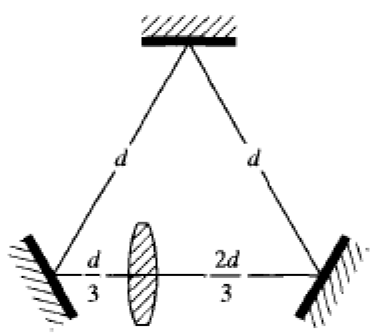
$$A = 1$$
$$B = 0$$
$$C = -\frac{j\lambda t^2}{2\pi}$$
$$D = 1$$

The required transmission ABCD matrix is :

$$\boxed{ABCD = \begin{bmatrix} 1 & 0 \\ -\frac{j\lambda t^2}{2\pi} & 1 \end{bmatrix}}$$

Problem 2.6 (Practice ABCD Law on Gaussian Beams in Optical Cavities)
Verdeyen Problem # 5.1.

5.1. Consider the optical cavity shown in the accompanying diagram. Assume that it is stable.



(a) Sketch an equivalent-lens waveguide.

(b) Compute the minimum spot size for a Gaussian beam and identify the place in the cavity where the beam achieves this minimum.

(1) Identify the appropriate unit cell that makes the $ABCD$ matrix symmetric.

(2) Identify the plane $z = 0$ in the cavity.

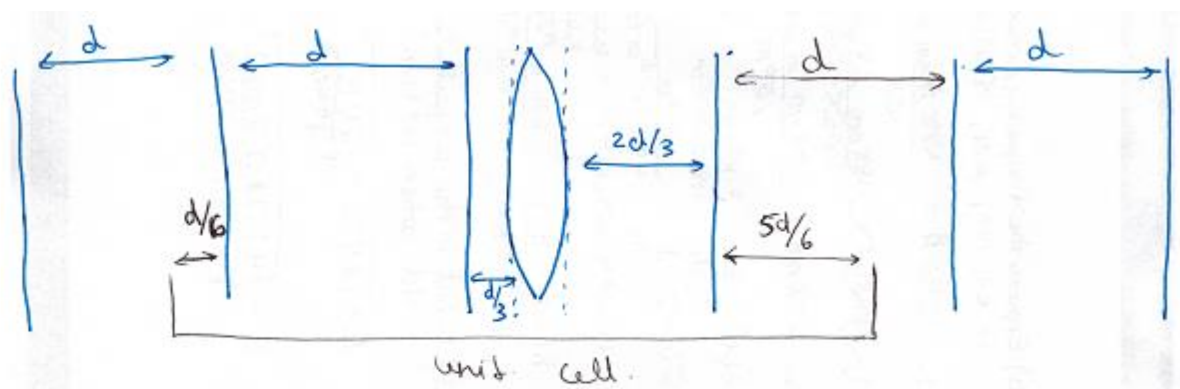
(3) What is the $ABCD$ matrix for the unit cell?

(c) What is the formula for the minimum spot size?

(d) Show that this formula is valid for stable cavities only.

Solution:

(a) The equivalent lens waveguide for the given setup: (Let the ray start just before the mirror on the top. Solid lines signify where the flat mirror is supposed to be.)



(b) 1.

The symmetric unit cell is marked on the above sketch

(b) 2.

From the sketch it can be seen that as it is symmetric, the beam waist or the position where the phase front is planar/$R = \infty$ – $\boxed{\text{plane of } z = 0 \text{ is at } \frac{5}{6}d \text{ from the flat mirror} \rightarrow \frac{3}{2}d \text{ from the lens.}}$

(b) 3.

We can define the path of the ray as – (i) distance $\frac{d}{6}$ in air + Flat mirror [ignored] + distance $\frac{4d}{3}$ in air, (ii) spherical lens of $'f'$, (iii) distance $\frac{2d}{3}$ in air + flat mirror [ignored] + distance $\frac{5d}{6}$ in air. Thus the ABCD matrix becomes,

$$ABCD = \begin{bmatrix} 1 & \frac{3d}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{3d}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3d}{2f} & \frac{3d}{2} \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3d}{2} \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 - \frac{3d}{2f} & \frac{3d}{2}\left(1 - \frac{3d}{2f}\right) \\ -\frac{1}{f} & 1 - \frac{3d}{2f} \end{bmatrix}}$$

(c) & (d)

Using the fact that $AD - BC = 1$, and equation 5.3.5, we get,

$$\frac{1}{q(z)} = \frac{A-D}{2B} - \frac{j\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}}{B}$$

$$= \frac{\left(1-\frac{3d}{2f}\right)-\left(1-\frac{3d}{2f}\right)}{3d\left(1-\frac{3d}{2f}\right)} - \frac{j\left[1-\left(\frac{\left(1-\frac{3d}{2f}\right)+\left(1-\frac{3d}{2f}\right)}{2}\right)^2\right]^{\frac{1}{2}}}{\frac{3d}{2}\left(1-\frac{3d}{2f}\right)}$$

$$= 0 - j\frac{\sqrt{1-\left(1-\frac{3d}{2f}\right)^2}}{\frac{3d}{2}\left(1-\frac{3d}{2f}\right)} = 0 - \frac{j\sqrt{1-A^2}}{B}$$

This makes sense only if the cavity is stable.

$$\frac{A+D}{2} < 1 \Rightarrow A < 1 \Rightarrow 1 - \frac{3d}{2f} < 1$$

Using the definition of stability,

$$AD - BC = 1 \Rightarrow BC = -(1-AD) = -(1-A^2)\ \ldots.[as\ A = D]$$

Using equation 5.3.7,

$$\frac{\pi n w^2(z_1)}{\lambda_0} = \frac{B}{\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}} = \frac{B}{\sqrt{1-A^2}} = -\frac{\frac{1}{C}(1-A^2)}{\sqrt{1-A^2}} = f\sqrt{1-\left(1-\frac{3d}{2f}\right)^2}$$

$$= f\sqrt{-\left(\frac{3d}{2f}\right)^2+\frac{3d}{f}} = f\sqrt{\frac{3d}{f}}\sqrt{1-\frac{3d}{4f}} = \sqrt{3df}\sqrt{1-\frac{3d}{4f}}$$

Spot size $w = \sqrt{-\frac{\lambda}{\pi}\left(\frac{1}{C}\right)\sqrt{1-A^2}}$

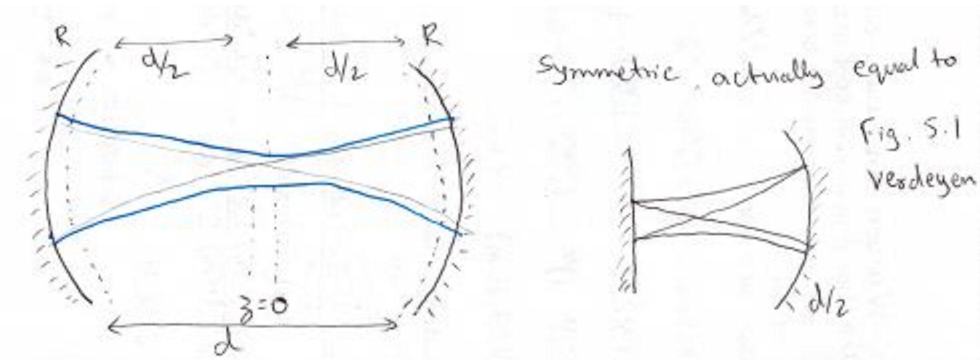
Problem 2.7 (Maximizing the Mode Volume)

Verdeyen Problem # 5.4.

5.4. **(a)** Construct a graph similar to Fig. 5.5 for the cavity of Fig. 5.2 ($R_1 = R_2$).
(b) Use the expansion law for a Gaussian beam to find the spot sizes on the mirrors.
(c) If $R_1 = R_2$ in Fig. 5.2, find the distance that maximizes the mode volume.

Solution:

(a) and (b) Cavity in Fig 5.2 when modified to make $R_1 = R_2 = R$, we get the following cavity:



Generally,

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$
$$w^2(z) = w_0^2\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$
$$z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

Forcing the phase surface to match the curved mirror at $z = \frac{d}{2}$,

$$R\left(\frac{d}{2}\right) = R = \frac{d}{2}\left[1 + \left(\frac{2z_0}{d}\right)^2\right] = \frac{d}{2}\left[1 + \frac{4z_0^2}{d^2}\right] = \frac{d}{2} + \frac{2z_0^2}{d}$$
$$z_0 = \sqrt{\left(R - \frac{d}{2}\right)\left(\frac{d}{2}\right)} = \sqrt{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)} = \frac{\pi w_0^2}{\lambda_0}$$
$$w_0^2 = \frac{\lambda_0 z_0}{\pi} = \frac{\lambda_0}{\pi}\sqrt{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)}$$
$$w^2(z) = \frac{\lambda_0 z_0}{\pi}\left[1 + \left(\frac{z_0}{z}\right)^2\right] = \frac{\lambda_0}{\pi}\sqrt{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)} \cdot \left[1 + \frac{z^2}{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)}\right]$$

At the mirror,

$$w^2\left(\frac{d}{2}\right) = \frac{\lambda_0}{\pi}\sqrt{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)}\left[1 + \frac{d^2}{\left(\frac{dR}{2}\right)\left(1 - \frac{d}{2R}\right)}\right]$$

Let $x = \dfrac{d}{2R}$

$$w^2\left(\frac{d}{2}\right) = \frac{\lambda_0}{\pi} R\sqrt{x}\sqrt{1-x}[1 + \frac{Rx}{1-x}]$$

At center,

$$w^2(0) = \frac{\lambda_0}{\pi} z_0 = \frac{\lambda_0}{\pi}\sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}} = \frac{\lambda_0}{\pi} R\sqrt{x}\sqrt{1-x}$$

We know,

$$w^2(z) = w_0^2\left[1+\left(\frac{z_0}{z}\right)^2\right] \qquad z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

$$w^2(z) = \frac{w_0^2}{z_0}\left[\frac{z_0^2+z^2}{z_0}\right] \qquad \frac{w_0^2}{z_0} = \frac{\lambda}{n\pi}$$

$$w^2(z) = \frac{\lambda_0}{\pi}\left[\frac{z_0^2+z^2}{z_0}\right] \qquad z_0 = \sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}}$$

$$w^2(z) = \frac{\lambda_0}{\pi}\left[\frac{\left(\left(\frac{dR}{2}\right)\left(1-\frac{d}{2R}\right)+z^2\right)}{\sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}}}\right]$$

At $\frac{d}{2}$,

$$w^2\left(\frac{d}{2}\right) = \frac{\frac{\lambda_0}{\pi}\left(\left(\frac{dR}{2}\right)\left(1-\frac{d}{2R}\right)+\frac{d^2}{4}\right)}{\sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}}}$$

Therefore, at either mirror, $R \to \pm\frac{d}{2}$, we have

$$w^2\left(\frac{d}{2}\right) = \frac{\lambda_0}{\pi}\frac{\sqrt{\frac{dR}{2}}}{\sqrt{1-\frac{d}{2R}}}$$

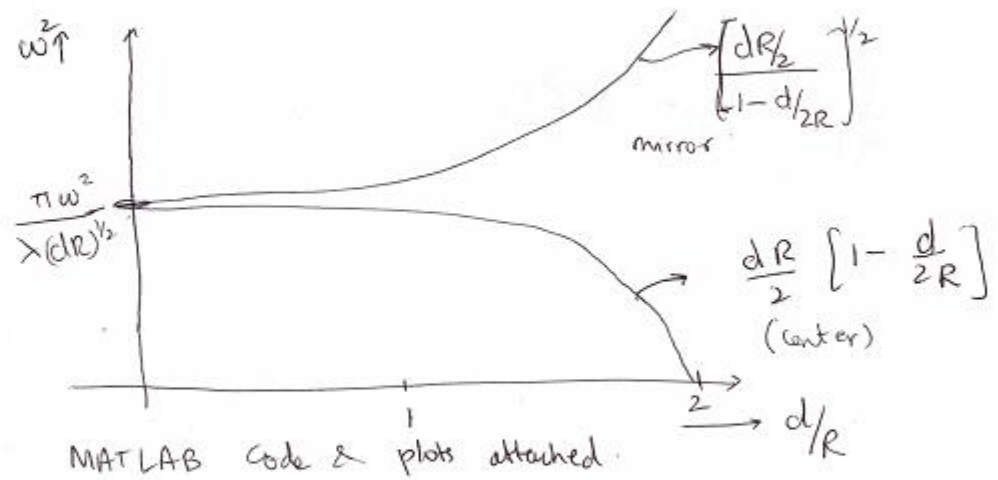
At $z = 0$,

$$w^2(0) = \frac{\lambda_0}{\pi} z_0 = \frac{\lambda_0}{\pi}\sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}}$$

Let $x = \frac{d}{2R}$,

$$\boxed{w^2(0) = \frac{\lambda_0 R}{\pi}\sqrt{x}\sqrt{1-x} \qquad w^2\left(\frac{d}{2}\right) = \frac{\lambda_0 R}{\pi}\frac{\sqrt{x}}{\sqrt{1-x}}}$$

Plotting these on MATLAB, we have the following sketch, (code and Plot attached)



$$w(z \gg z_0) = \frac{w_0 z}{z_0} = \frac{\lambda_0 z}{\pi n w_0}$$

On the mirror, $z = \frac{d}{2}$, spot size is,

$$\boxed{w = \frac{\lambda d}{2\pi n w_0} \rightarrow w\left(\frac{d}{2}\right) = \left(\frac{\lambda_0}{\pi}\frac{\sqrt{\frac{dR}{2}}}{\sqrt{1-\frac{d}{2R}}}\right)^{\frac{1}{2}}}$$

(c)

$$E_0^2 V = \int_0^d \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x,y,z)E^*(x,y,z)\, dx\, dy\, dz$$

$$E_0^2 V_{m,p} = E_0^2 \int_0^d \frac{w_0^2}{w^2(z)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_m^2\left(\frac{\sqrt{2}x}{w}\right) \exp\left(-\frac{2x^2}{w^2}\right). H_p^2\left(\frac{\sqrt{2}y}{w}\right) \exp\left(-\frac{2y^2}{w^2}\right) dx\, dy\, dz$$

$$V_{m,p} = \int_0^d \frac{w_0^2}{w^2(z)} dz \int_{-\infty}^{\infty} H_m^2(u) \exp(-u^2)\, du\, . \int_{-\infty}^{\infty} H_p^2(u) \exp(-u^2)\, du$$

$$= \int_0^d \frac{w_0^2}{w^2(z)} dz \left[2^m m!\, \pi^{\frac{1}{2}}\right]\left[2^p p!\, \pi^{\frac{1}{2}}\right] = \pi w_0^2 d\; m!\, p!\, 2^{m+p} = A w_0^2 d$$

$$w_0^2 = \frac{\lambda_0}{\pi}\sqrt{\frac{dR}{2}}\sqrt{1-\frac{d}{2R}}$$

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$$V_{m,p} = A'\sqrt{\frac{dR}{2}}\left(\sqrt{1-\frac{d}{2R}}\right)d = A'\left[\left(\frac{dR}{2}\right)(d^2)\left(1-\frac{d}{2R}\right)\right]^{\frac{1}{2}} = A'^{\left[\frac{R}{2}d^3-\frac{1}{4}d^4\right]^{\frac{1}{2}}}$$

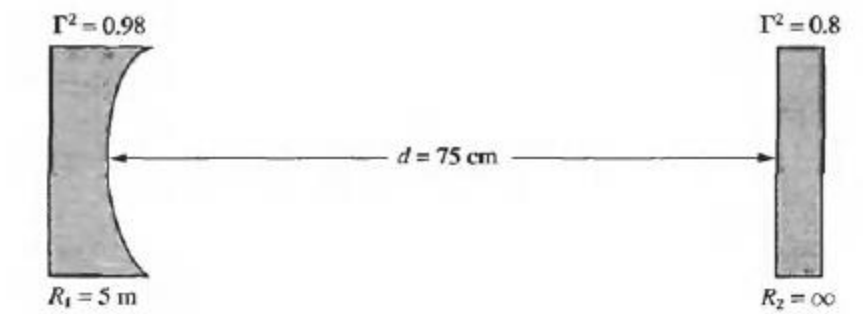
$$A' = \frac{\lambda_0}{\pi}\left(\frac{\pi}{2}\right) m!\,p!\,2^{m+p}$$

$$\frac{\partial V}{\partial d} = \frac{A'}{2}\left[\frac{R}{2}d^3-\frac{1}{4}d^4\right]^{-\frac{1}{2}}\left[\frac{3R}{2}d^2-d^3\right]|_{dmax} = 0$$

$$\boxed{d = \frac{3R}{2} \text{ maximizes the mode volume}}$$

Problem 2.8 (Spot size, Photon Lifetime, and Quality Factor of a CO_2 Laser)
Verdeyen Problem # 5.11.

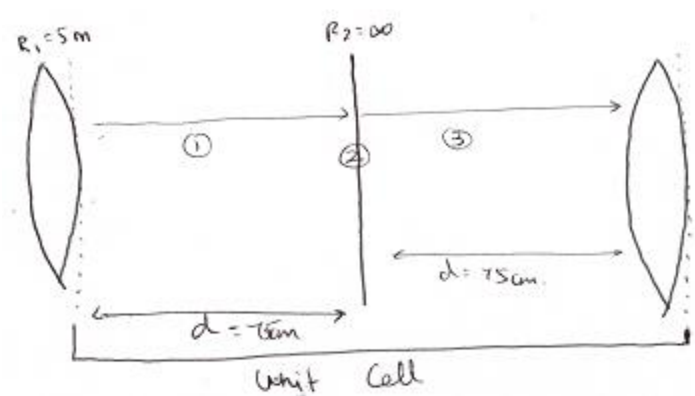
5.11. A CO_2 laser operating at $\lambda_0 = 10.6\ \mu m$ uses the following optical cavity.



(a) Use the $ABCD$ law to find an expression and numerical value for the spot size on the spherical mirror.
(b) Evaluate the photon lifetime and Q of the cavity.
(c) Sketch the intensity (dot) pattern of a $TEM_{2,3}$ mode.

Solution:

(a) Re-drawing the given arrangement to show the ray propagation and the unit cell for the cavity, we get the following (Let the ray start just after the spherical mirror):



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Ray path → 1- over a distance $'d'$ in air to the plane mirror; 2- plane mirror reflection; 3- over a distance $'d'$ in air to the spherical mirror

$$ABCD = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2d \\ -\frac{2}{R_1} & 1-\frac{4d}{R_1} \end{bmatrix}$$

Using the equation for spot size from section 5.3 which uses application of stability conditions to the $ABCD$ matrix. Setting the q parameter at $z1$ equal to itself transformed by the ABCD matrix:

$$\frac{\pi n \omega^2(z_1)}{\lambda_0} = \frac{B}{\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}} = \frac{2d}{\left[1-\left(\frac{1+1-\frac{4d}{R_1}}{2}\right)^2\right]^{\frac{1}{2}}} = \frac{2d}{\left[1-\left(1-\frac{2d}{R_1}\right)^2\right]^{\frac{1}{2}}}$$

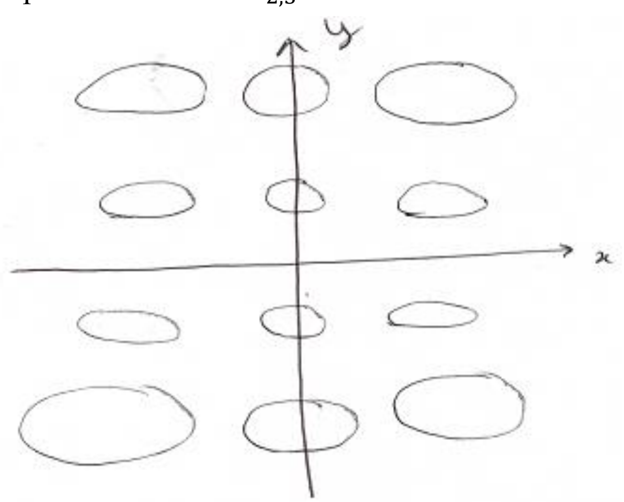
$$\omega(z_1) = \sqrt{\left(\frac{\lambda}{\pi}\right)\left(\frac{2d}{\left[1-\left(1-\frac{2d}{R_1}\right)^2\right]^{\frac{1}{2}}}\right)} = \sqrt{\left(\frac{10.6\mu m}{\pi}\right)\left(\frac{2\times 75cm}{\left[1-\left(1-\frac{2(75cm)}{5m}\right)^2\right]^{\frac{1}{2}}}\right)} = \boxed{0.266\text{ cm}}$$

(b) To find the photon lifetime and Q, we can use the following equations from section 6.4:

$$\tau_p = \frac{\left(\frac{2nd}{c}\right)}{1-\Gamma_1^2\Gamma_2^2} = \frac{\left(\frac{2\times 1\times 75\text{cm}}{3\times 10^8\text{m/s}}\right)}{1-(0.98^2)(0.8^2)} = \boxed{2.31\times 10^{-8}\text{s}}$$

$$Q = \omega\tau_p = \frac{2\pi c}{\lambda_0}\tau_p = \frac{(2\pi\times 3\times 10^8\text{m/s})}{10.6\mu m}(2.31\times 10^{-8}) = \boxed{4.12\times 10^6}$$

(c) Dot pattern of the $\text{TEM}_{2,3}$ mode:



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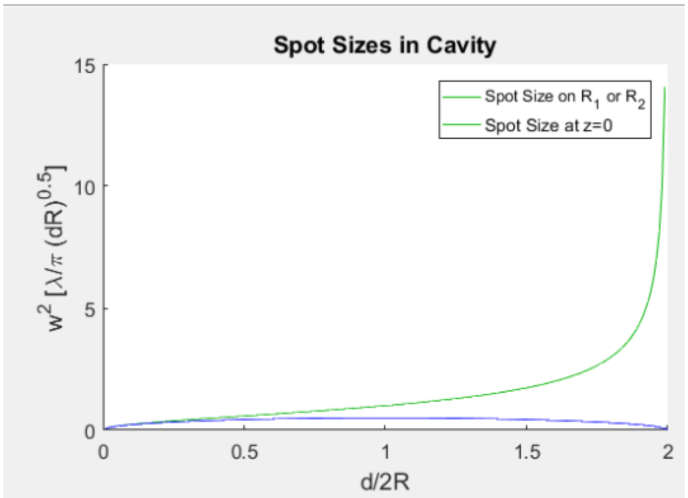
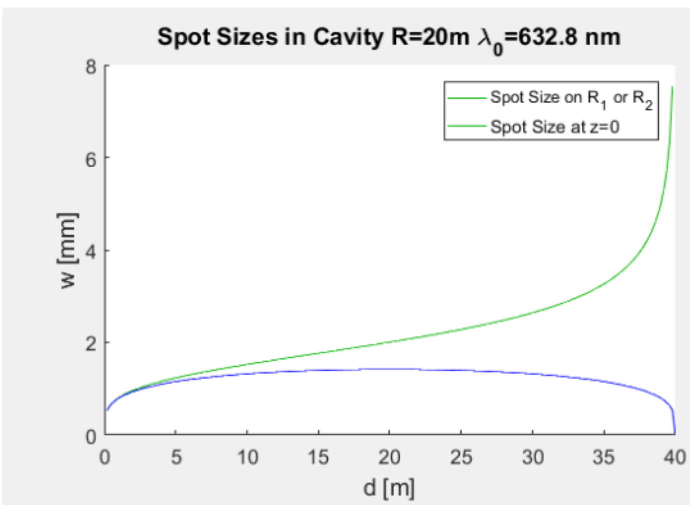
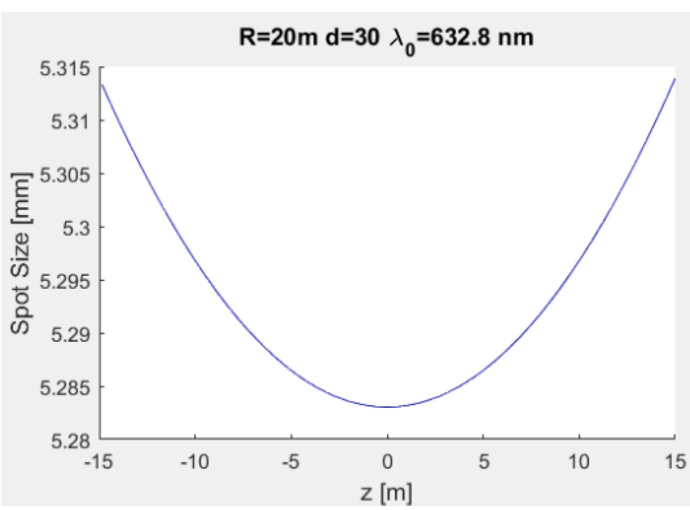
Problem 2.9

Mathematica Notebook Attached

Rest of Plots for Problem 2.7







Here is a copy of a Mathematica program I wrote to make the plot. Undoubtedly, each of you has a different scheme. So long as you were able to compute the ray positions and make a plot, you were graded full credit.

In[97]:=

```
count = 12;
n = 1;
r1 = 204;
r2 = 323.;
l = 500.;
m1 = {{1, 0}, {-2 / r1, 1}};
m2 = {{1, 0}, {-2 / r2, 1}};
length = {{1, l}, {0, 1}};
pos1 = {10, 0};
p = {{0, pos1[[1]]}};
For[n = 0, n < count, n++,
  pos2 = length.m2.pos1;
  AppendTo[p, {1, pos2[[1]]}];
  pos1 = length.m1.pos2;
  AppendTo[p, {0, pos1[[1]]}]];
Graphics[Line[p]]
```

Out[108]=