ECE 4300: Lasers and Optoelectronics
Fall 2016, Debdeep Jena and Clif Pollock
Solutions to Assignment 3: By Kevin Lee

Problem 3.1 (Verdeyen 5.13)

First, I calculate the ABCD matrix for beam traveling through the lens and space.

$$\mathrm{T}=\begin{pmatrix}1 & d_1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & d_2\\0 & 1\end{pmatrix}=\begin{pmatrix}1-\frac{d_2}{f} & d_2+d_1(1-\frac{d_2}{f})\\ \frac{-1}{f} & 1-\frac{d_1}{f}\end{pmatrix}$$

According to ABCD law, we can have the following relation for beam waist.

$$q_2=\frac{Aq_1+B}{Cq_1+D}$$

$$q_2(z)=z+jz_{02}$$

$$q_1(z)=z+jz_{01}$$

Since it requires mode-matching, the two minimum beam waists should satisfy the ABCD law. So we have

$$q_2(0)=\frac{Aq_1(0)+B}{Cq_1(0)+D}=\frac{A\times jz_{01}+B}{C\times jz_{01}+D}=jz_{02}$$

$$A\times jz_{01}+B=D\times jz_{02}-z_{01}z_{02}C$$

$$\mathrm{B}=-z_{01}z_{02}C$$

$$\mathrm{A}z_{01}=Dz_{02}$$

$$-\frac{B}{C}=z_{01}z_{02}$$

$$\frac{A}{D}=\frac{z_{02}}{z_{01}}$$

Substitute the above ABCD into the equations.

$$z_{01}z_{02}=fd_2+d_1(f-d_2)$$

$$\frac{z_{02}}{z_{01}}=\frac{1-\frac{d_2}{f}}{1-\frac{d_1}{f}}=\frac{f-d_2}{f-d_1}$$

$$-(f-d_1)\frac{z_{02}}{z_{01}}+f=d_2$$

$$z_{01}z_{02}=f\left(f-(f-d_1)\frac{z_{02}}{z_{01}}\right)+d_1\left(f-\left(f-(f-d_1)\frac{z_{02}}{z_{01}}\right)\right)$$

$$=f\left(f-f\frac{z_{02}}{z_{01}}\right)+f\frac{z_{02}}{z_{01}}d_1+d_1(f-d_1)\frac{z_{02}}{z_{01}}$$

$$f\left(f-f\frac{z_{02}}{z_{01}}\right)+2f\frac{z_{02}}{z_{01}}d_1-\frac{z_{02}}{z_{01}}d_1{}^2=z_{01}z_{02}$$

$$\frac{z_{02}}{z_{01}}d_1{}^2-\left(2f\frac{z_{02}}{z_{01}}\right)d_1+z_{01}z_{02}-f\left(f-f\frac{z_{02}}{z_{01}}\right)=0$$

$$d_1 = \frac{2f\frac{z_{02}}{z_{01}} \pm \sqrt{4(f\frac{z_{02}}{z_{01}})^2 - 4\frac{z_{02}}{z_{01}} \times (z_{01}z_{02} - f\left(f - f\frac{z_{02}}{z_{01}}\right))}}{2\frac{z_{02}}{z_{01}}}$$

$$= f \pm \sqrt{f^2 - \frac{z_{01}}{z_{02}} \times (z_{01}z_{02} - f\left(f - f\frac{z_{02}}{z_{01}}\right))}$$

$$= f \pm \sqrt{f^2 - ((z_{01})^2 - f^2(\frac{z_{01}}{z_{02}} - 1))}$$

$$= f \pm \sqrt{f^2 - (z_{01})^2 + f^2(\frac{z_{01}}{z_{02}} - 1)} = f \pm \sqrt{\frac{z_{01}}{z_{02}}f^2 - (z_{01})^2}$$

$$d_2 = f - \frac{z_{02}}{z_{01}}(f - d_1) = f - \frac{z_{02}}{z_{01}}\left(f - \left(f \pm \sqrt{\frac{z_{01}}{z_{02}}f^2 - (z_{01})^2}\right)\right)$$

$$= f - \frac{z_{02}}{z_{01}}\left(\mp\sqrt{\frac{z_{01}}{z_{02}}f^2 - (z_{01})^2}\right) = f \pm \frac{z_{02}}{z_{01}}\sqrt{\frac{z_{01}}{z_{02}}f^2 - (z_{01})^2}$$

Problem 3.2 (Verdeyen 6.1)

$$d = \frac{3}{4} R_2$$

$$\Gamma_1^2 = 0.99$$

$$\Gamma_2^2 = 0.97$$

(a)

From (6.5.3), we have the resonant frequency.

$$\nu_{m,p,q} = \frac{c}{2nd}[q + \frac{1+m+p}{\pi} cos^{-1}(1-\frac{d}{R_2})^{1/2}]$$

$$\nu_{m,p,q} = \frac{c}{2nd}[q + \frac{1+m+p}{3}]$$

Now we're considering $\mathrm{TEM}_{0,0}$.

$$\nu_{0,0,q} = \frac{c}{2nd}[q + \frac{1}{3}]$$

(b)

(1) FSR

$$R_2 = 2m$$

$$d = \frac{3}{2} m$$

$$\lambda = 5000\text{Å} = 500nm$$

$$FSR = \frac{c}{2nd} = \frac{3 \times 10^8 m/s}{2 \times \frac{3}{2} m} = 1 \times 10^8 s^{-1} = 100MHz$$

$$Light\ frequency = \frac{c}{\lambda} = 6 \times 10^{14} Hz$$

$$\frac{FSR}{Light\ frequency} = \frac{\Delta\lambda}{\lambda} = \frac{1 \times 10^8}{6 \times 10^{14}}$$

$$\Delta\lambda = 5000\text{Å} \times \frac{1 \times 10^8}{6 \times 10^{14}} = 8.33 \times 10^{-4}\text{Å}$$

(2) Cavity Q

From (6.3.5), we have

$$Q = \frac{4\pi nd}{\lambda}\left(\frac{1}{1-R_1R_2}\right) = 9.496 \times 10^8$$

(3) Photon lifetime

From (6.4.2b), we have

$$\tau_p = \frac{2nd/c}{1-R_1R_2} = 2.5189 \times 10^{-7} s = 251.89nsec$$

(4) Finess

From (6.3.8), we have

$$F = \frac{2\pi}{1 - R_1 R_2} = 158.267$$

Problem 3.3 (Verdeyen 6.5 – 6.10)

6.5

$$\nu_0 = 5 \times 10^{14} Hz$$

$$\lambda_0 = \frac{c}{\nu_0} = 600nm$$

6.6

$$FSR = 125MHz = \frac{c}{2nd}$$

$$d = 1.2m$$

6.7

$$F = \frac{FSR}{\Delta\nu_{1/2}} = \frac{125MHz}{2.5MHz} = 50$$

6.8

$$Q = \frac{\nu_0}{\Delta\nu_{1/2}} = \frac{5 \times 10^{14} Hz}{2.5MHz} = 2 \times 10^8$$

6.9

From (6.4.4), we have photon lifetime.

$$\tau_p = \frac{Q}{2\pi\nu_0} = \frac{2 \times 10^8}{2\pi \times 5 \times 10^{14} Hz} = 63.662nsec$$

6.10

From p.154, the requirement of a laser to oscillate is its lifetime is negative, meaning photons are growing in the cavity.

From (6.3.8), we can calculate the $R_1 R_2$.

$$F = \frac{2\pi}{1 - R_1 R_2}$$

$$1 - R_1 R_2 = \frac{2\pi}{F}$$

$$R_1 R_2 = 1 - \frac{2\pi}{F} = 0.8743$$

$$\tau_p = \frac{2nd/c}{1 - GR_1 GR_2} < 0$$

$$1 - GR_1 GR_2 < 0$$

$$G > \frac{1}{\sqrt{R_1 R_2}} = 1.06947$$

Problem 3.4 (Verdeyen 6.19)

(a)

$$T = \begin{pmatrix} 1 & 0 \\ \frac{-2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0.75 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0.75 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ \frac{-2}{3} & 0 \end{pmatrix}$$

$$0 \le \frac{A + D + 2}{4} = \frac{3}{4} \le 1$$

It's a stable cavity.

(b)

$$\nu_{m,p,q} = \frac{c}{2nd}[q + \frac{1 + m + p}{\pi} cos^{-1}(1 - \frac{d}{R_2})^{1/2}]$$

$$\frac{d}{R_2} = \frac{0.75}{3}$$

$$\nu_{m,p,q} = \frac{c}{2nd}[q + \frac{1 + m + p}{\pi} \times 0.5236]$$

For $TEM_{0,0,q}$, its frequency is

$$\nu_{0,0,q} = \frac{c}{2nd}[q + \frac{1}{\pi} \times 0.5236]$$

For $TEM_{1,0,q}$, its frequency is

$$\nu_{1,0,q} = \frac{c}{2nd}[q + \frac{2}{\pi} \times 0.5236]$$

$$\nu_{1,0,q} - \nu_{0,0,q} = \frac{c}{2nd} \times \frac{1}{\pi} \times 0.5236 = 33.3MHz$$

(c)

From (6.6.2a), the $TEM_{0,0,q}$ transmission is

$$T = 1 - e^{-2(\frac{a}{\omega})^2}$$

Less than 0.1% loss, which means $T = 99.9\% = 0.999$

$$T = 1 - e^{-2\left(\frac{a}{\omega}\right)^2} = 0.999$$

$$\frac{a}{\omega} = 1.85846$$

The beam waist is largest at the spherical mirror.

From (5.3.9), we have the beam waist.

$$\frac{\pi\omega^2}{\lambda} = \sqrt{dR}\sqrt{1 - \frac{d}{R}}$$

$$\omega = 0.498mm$$
$$a = 0.9256mm$$
$$Diameter = 2a = 1.85mm$$

(d)

From (8.1.2), we have the condition to overcome the loss and induce oscillation in the cavity.

$$\gamma_0(\nu) \geq \frac{1}{2l_g}\ln(\frac{1}{R_1R_2})$$

$$\frac{1}{2l_g}\ln\left(\frac{1}{R_1R_2}\right) = \frac{1}{0.5}\ln\left(\frac{1}{1 \times 0.95}\right) = 0.0512933m^{-1} = 5.13 \times 10^{-4}cm^{-1}$$

Problem 3.5 (Verdeyen 6.25)

(a)

From (3.5.1), we know the phase shift of a mode inside the cavity.

$$kd - (1 + m + p)tan^{-1}\left(\frac{d}{z_0}\right) = q\pi$$

But this equation is for one flat mirror and a curved mirror cavity. Here we have two flat mirrors and a lens in between. We need to modify the equation for phase matching.

$$kd_1 - (1 + m + p)tan^{-1}\left(\frac{d_1}{z_{01}}\right) + kd_2 - (1 + m + p)tan^{-1}\left(\frac{d_2}{z_{02}}\right) = q\pi$$

$$k = \frac{2\pi}{\lambda/n} = \frac{2\pi n}{c/\nu} = \frac{2\pi\nu n}{c}$$

$$\nu = \frac{c}{2(d_1 + d_2)}\left(q + (1 + m + p)(tan^{-1}\left(\frac{d_1}{z_{01}}\right) + tan^{-1}\left(\frac{d_2}{z_{02}}\right))\right)$$

(b)

From (6.4.2b), we have photon lifetime.

$$\tau_p = \frac{2n(d_1 + d_2)/c}{1 - TR_1TR_2} = 28.18\mathrm{nsec}$$

(c)

$$\lambda_0 = 5145\text{Å}$$

From (6.3.5), we have the quality factor.

$$Q = \frac{4\pi n(d_1 + d_2)}{\lambda_0}\left(\frac{1}{1 - TR_1TR_2}\right) = 1.032 \times 10^8$$

(d)

$$\tau_p = \frac{2n(d_1 + d_2)/c}{1 - GTR_1GTR_2} = -99.4\mathrm{nsec}$$

From (6.4.5), linewidth can be obtained.

$$\Delta\nu_{1/2} = \frac{1}{2\pi\tau_p} = 5.648MHz$$

Problem 3.6 (Verdeyen 7.14)

(a)

$$g_1 = g_2$$
$$\mathrm{n} = 1$$

(7.3.4)

At equilibrium, the time rate of change should be zero.

$$\frac{N_2}{N_1} = \frac{B_{12}\rho(\nu)}{A_{21} + B_{21}\rho(\nu)} = e^{-h\nu/kT}$$

Neglect the stimulated emission process, which means $B_{21} = 0$.

$$\frac{B_{12}\rho(\nu)}{A_{21}} = e^{-h\nu/kT}$$

$$\rho(\nu) = \frac{A_{21}}{B_{12}} e^{-h\nu/kT}$$

According to Einstein's theory, we can get the coefficient.

$$\frac{A_{21}}{B_{21}} = \frac{A_{21}}{B_{12}} = \frac{8\pi n^2 n_g h\nu^3}{c^3}$$

$$\rho(\nu) = \frac{8\pi n^2 n_g h\nu^3}{c^3} e^{-h\nu/kT}$$

This is the low temperature approximation for the original blackbody radiation.

$$\frac{1}{e^{\frac{h\nu}{kT}} - 1} \sim e^{-h\nu/kT}$$

when $e^{\frac{h\nu}{kT}} \gg 1$, it means T is small.

(b) Similar to the argument in the previous problem. If $\frac{h\nu}{kT} \gg 1$, then it automatically satisfies that $e^{\frac{h\nu}{kT}} \gg 1$. And it leads to

$$\frac{1}{e^{\frac{h\nu}{kT}} - 1} \sim e^{-h\nu/kT}$$

We have the energy density in the previous problem.

$$\rho(\nu) = \frac{A_{21}}{B_{12}} e^{-h\nu/kT}$$

In order to match the results, we require

$$\frac{A_{21}}{B_{12}} = \frac{8\pi n^2 n_g h\nu^3}{c^3} = \frac{8\pi h\nu^3}{c^3}$$

with $\mathrm{n} = 1$ and $n_g \sim n$.

(c)

From the above derivation, we have the Wein's distribution.

$$\rho_{Wein}(\nu) = \frac{8\pi h\nu^3}{c^3} e^{-h\nu/kT}$$

From (7.2.5), we have Rayleigh-Jeans distribution.

$$\rho_{\text{Rayleigh-Jeans}}(\nu) = \frac{8\pi\nu^2}{c^3} kT$$

And Planck's distribution is the following.

$$\rho_{\text{Planck}}(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

By observations, the Wein and Rayleigh-Jeans are both approximations under extreme conditions.

For Wein's distribution, we have shown that it is an extreme case when $\frac{h\nu}{kT} \gg 1$.

For Rayleigh-Jeans distribution, it is the other condition $\frac{h\nu}{kT} \ll 1$.

$$\rho_{\text{Planck}}(\nu) = \frac{8\pi\nu^2 kT}{c^3} \frac{h\nu/kT}{e^{\frac{h\nu}{kT}} - 1}$$

Note that $\lim_{x\to 0} \frac{x}{e^x - 1} = 1$. Then we have the Rayleigh-Jeans distribution

$\lim_{\frac{h\nu}{kT} \ll 1} \rho_{\text{Planck}}(\nu) = \frac{8\pi\nu^2}{c^3} kT.$

Problem 3.7 (Verdeyen 7.4)

(a)

First we need to write down the energy (here we're using frequency because frequency is proportional to energy) distribution function of each group of atoms. As described in the problem, we should use Lorentzian function. The lineshape function can be written down in the following equation.

$$g(\nu) = \int_{-\infty}^{\infty} p(f)df \times \frac{\Delta\nu_h}{2\pi[(f-\nu)^2 + (\Delta\nu_h/2)^2]}$$

I ignore the Planck constants here because we only concern about the shape, not the exact number.

From the figure of p(f), we can modify the above lineshape function.

$$g(\nu) = \int_{\nu_0-\Delta\nu_s/2}^{\nu_0+\Delta\nu_s/2} \frac{1}{\Delta\nu_s} df \frac{\Delta\nu_h}{2\pi[(f-\nu)^2 + (\Delta\nu_h/2)^2]}$$

$$= \frac{\Delta\nu_h}{2\pi\Delta\nu_s}\int_{\nu_0-\Delta\nu_s/2}^{\nu_0+\Delta\nu_s/2} \frac{df}{[(f-\nu)^2 + (\Delta\nu_h/2)^2]}$$

Let $x = f - \nu$

$$g(\nu) = \frac{\Delta\nu_h}{2\pi\Delta\nu_s}\int_{\nu_0-\frac{\Delta\nu_s}{2}+\nu}^{\nu_0+\frac{\Delta\nu_s}{2}+\nu} \frac{dx}{[x^2 + (\Delta\nu_h/2)^2]}$$

Note:

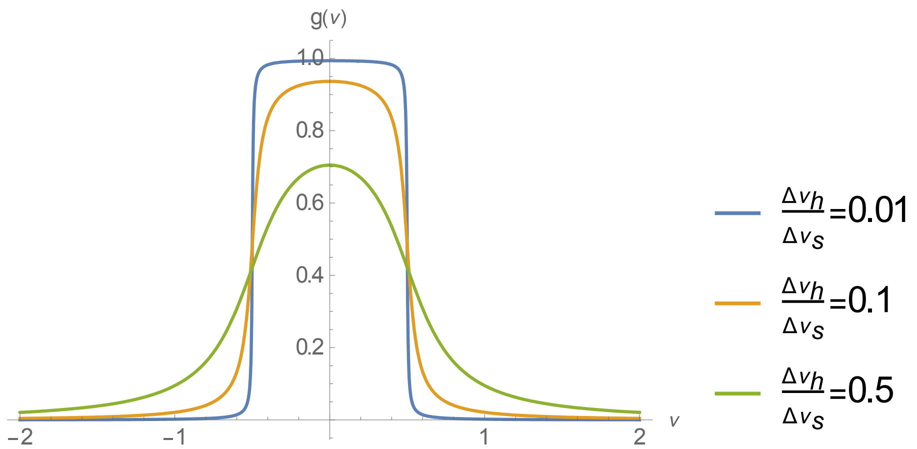
$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$$

$$g(\nu) = \frac{\Delta\nu_h}{2\pi\Delta\nu_s} \times \frac{2}{\Delta\nu_h}\left[tan^{-1}\left(\frac{2}{\Delta\nu_h}\left(\nu_0 + \nu + \frac{\Delta\nu_s}{2}\right)\right) - tan^{-1}\left(\frac{2}{\Delta\nu_h}(\nu_0 + \nu - \frac{\Delta\nu_s}{2})\right)\right]$$

(b)

Without loss of generality, I am going to assume the following parameters for the plot.

$$\Delta\nu_s = 1 \; and \; \nu_0 = 0$$



Problem 3.8 (Verdeyen 7.10)

(a)

$$\frac{dN_2}{dt} = P_2 - \frac{N_2}{\tau_2}$$

$$\frac{dN_1}{dt} = \frac{N_2}{\tau_2} - \frac{N_1}{\tau_1}$$

(b)

Solve the differential equation by Mathematica.

$$N_2(t) = P_2\tau_2(1 - e^{-\frac{t}{\tau_2}})$$

$$\frac{dN_1}{dt} = P_2\left(1 - e^{-\frac{t}{\tau_2}}\right) - \frac{N_1}{\tau_1}$$

$$N_1(t) = P_2\tau_1(1 - \frac{\tau_1}{\tau_1 - \tau_2}e^{-\frac{t}{\tau_1}} + \frac{\tau_2}{\tau_1 - \tau_2}e^{-\frac{t}{\tau_2}})$$

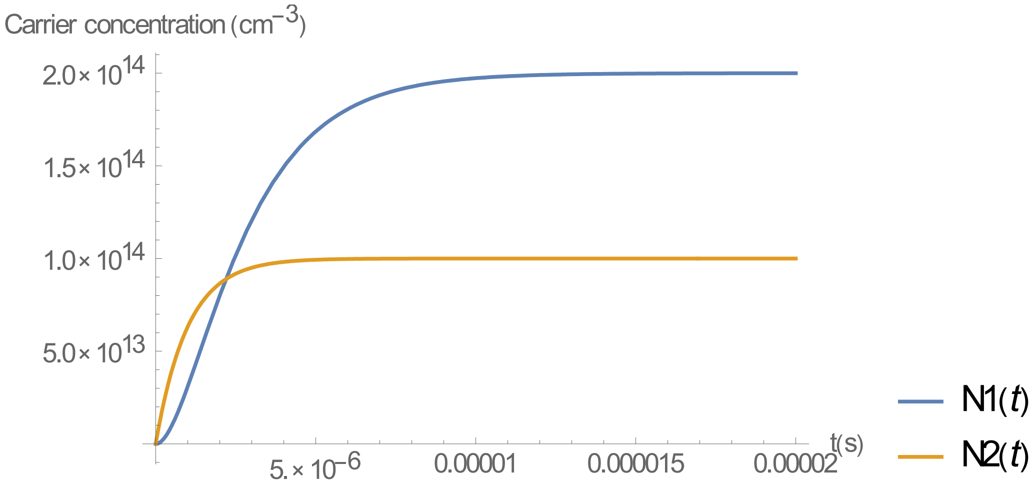
(c)

We have the lifetime of each state and the pumping strength.

$$\tau_1 = 2\mu s$$

$$\tau_2 = 1\mu s$$

$$P_2 = 10^{20} cm^{-3} s^{-1}$$



Inversion happens when N2 state has more carriers, which means $N_2 > N_1$. I used Mathematica to get a numerical solution.

$$\delta t = 2.1972 \mu s$$

(d)

Steady state means $t \to \infty$.

$$\lim_{t\to\infty} N_1(t) = P_2\tau_1 = 2 \times 10^{14} cm^{-3}$$

$$\lim_{t\to\infty} N_2(t) = P_2\tau_2 = 1 \times 10^{14} cm^{-3}$$