ECE 4300: Lasers and Optoelectronics
Fall 2016, Debdeep Jena and Clif Pollock, Cornell University
Solutions to Assignment 4: By Kevin Lee and Stephanie Sanders

Problem 4.1 (Verdeyen Problem #8.7)

(a)

From (7.4.7), simulated emission cross section is defined as following.

$$\sigma(\nu) = A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu)$$

$$A_{21} = 6 \times 10^6 sec^{-1}$$

From the figure, the emission is state 2 to state 1, which has a transition energy of 5.5eV – 3.2eV = 2.3eV.

$$\lambda = \frac{1.24eV}{Eg(eV)} = \frac{12.4}{2.3} = 0.539\mu m = 539nm$$

$$g(\nu)\ is\ the\ lineshape\ function, which\ is\ Lorentzian\ here.$$

$$g(\nu) = \frac{\Delta\nu}{2\pi((\nu-\nu_0)^2 + (\Delta\nu/2)^2)}$$

$$\Delta\nu = 10\text{GHz}$$

$$\sigma(\nu_0) = 6 \times 10^6 sec^{-1} \times \frac{(539nm)^2}{8\pi} \times \frac{2}{\pi \times 10GHz} = 4.415 \times 10^{-14}cm^2$$

(b)

From (7.5.2), we have the laser gain equation.

$$\gamma_0(\nu) = \sigma(\nu)(N_2 - \frac{g_2}{g_1}N_1)$$

Here we assume

$$N_1 \approx 0$$

The small signal gain coefficient is

$$\gamma_0(\nu_0) = \sigma(\nu_0) \times N_2 = 0.01cm^{-1}$$

$$N_2 = 2.265 \times 10^{11}cm^{-3}$$

Under steady state condition, the state 2's rate equation is following.

$$\frac{dN_2}{dt} = 0 = R_2 - N_2(\frac{1}{\tau_{21}} + \frac{1}{\tau_{20}})$$

$$R_2 = N_2 \times (\frac{1}{\tau_{21}} + \frac{1}{\tau_{20}}) = 3.4 \times 10^{18}s^{-1}cm^{-3}$$

(c)

From (8.3.7), saturation intensity is defined as

$$I_s = \frac{h\nu}{\sigma\tau_2}$$

To find the lifetime at state 2, the system is at equilibrium.

$$\frac{1}{\tau_2} = \frac{1}{\tau_{21}} + \frac{1}{\tau_{20}} = \frac{1}{100ns} + \frac{1}{200ns} = 6.67 \times 10^{-8}s$$

$$I_s = \frac{2.3eV}{4.415 \times 10^{-14}cm^2 \times 6.67 \times 10^{-8}s} = 125.017\ Wcm^{-2}$$

(d)

To achieve the desired carrier concentration at state 2, we need pumping transition of 5.5eV.

$$Pump\ power = 5.5\text{eV} \times R_2 = 2.99Wcm^{-3}$$

(e)

The Q-factor of this system should not change between units. So we have the following relation.

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\nu}{\nu}$$

$$\nu = \frac{c}{\lambda} = 5.57 \times 10^{12}Hz$$

$$\Delta\lambda = \frac{10GHz}{5.57 \times 10^{14}Hz} * 5390\text{Å} = 0.0968\text{Å}$$

Wavenumber is defined as the inverse of wavelength.

$$\bar{\nu} = \frac{1}{\lambda} = 18552.9cm^{-1}$$

$$\Delta\bar{\nu} = \frac{10GHz}{5.57 \times 10^{14}Hz} * 18552.9cm^{-1} = 0.333cm^{-1}$$

Problem 4.2 (Verdeyen Problem #8.11)

(a)

We can apply the procedure in section 8.3 to solve this problem. But in this problem, there is no transition specified from state 2 to state 0. I will assume the time constant is infinitely large, which means there is no transition from state 2 to 0.

Assume the pumping rate is a step function and there is no stimulated emission. Hence, (8.3.2a) and (8.3.2b) can be expressed as following.

$$\frac{dN_2}{dt} = R_2(t) - \frac{N_2}{\tau_2}$$

$$\frac{dN_1}{dt} = R_1(t) + \frac{N_2}{\tau_{21}}$$

Solve these two differential equations, we get

$$N_2(t) = R_{20}\tau_2(1 - e^{\frac{-t}{\tau_2}})$$

$$N_1(t) = \frac{\mathrm{e}^{-t/\tau_1}}{\tau_{21}}\int_0^t N_2(t')e^{\frac{t'}{\tau_1}}dt'$$

For $t \geq \tau_1$, we have (8.3.4c).

$$N_1(t) = \phi_{21}R_{20}\tau_1\{1 + \frac{\frac{\tau_1}{\tau_2}}{1-\frac{\tau_1}{\tau_2}}e^{-\frac{t}{\tau_1}} - \frac{1}{1-\frac{\tau_1}{\tau_2}}e^{-\frac{t}{\tau_2}}\}$$

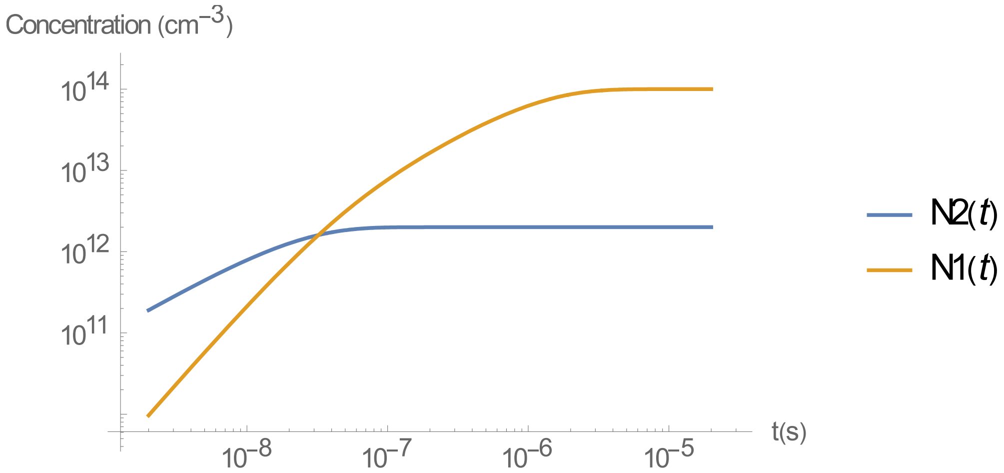
For equilibrium populations, meaning $t \to \infty$, $N_2(t) \to R_{20}\tau_2 = 2 \times 10^{12} cm^{-3}$ and $N_1(t) \to R_{20}\tau_1 = 1 \times 10^{14} cm^{-3}$.

(b)

It's obvious that this system is not suitable system for a CW laser. For a laser operation, we need population inversion, which means $N_2 > N_1$. For pulsed laser, we can tolerate that the system only has certain limited time of population inversion. However, CW laser requires the system to be always in population inversion as the laser is turned on. Continuous wave operation means the system is always lasing, which is always in population inversion.

(c)

Plot above two functions in Mathematica in Log Log scale.



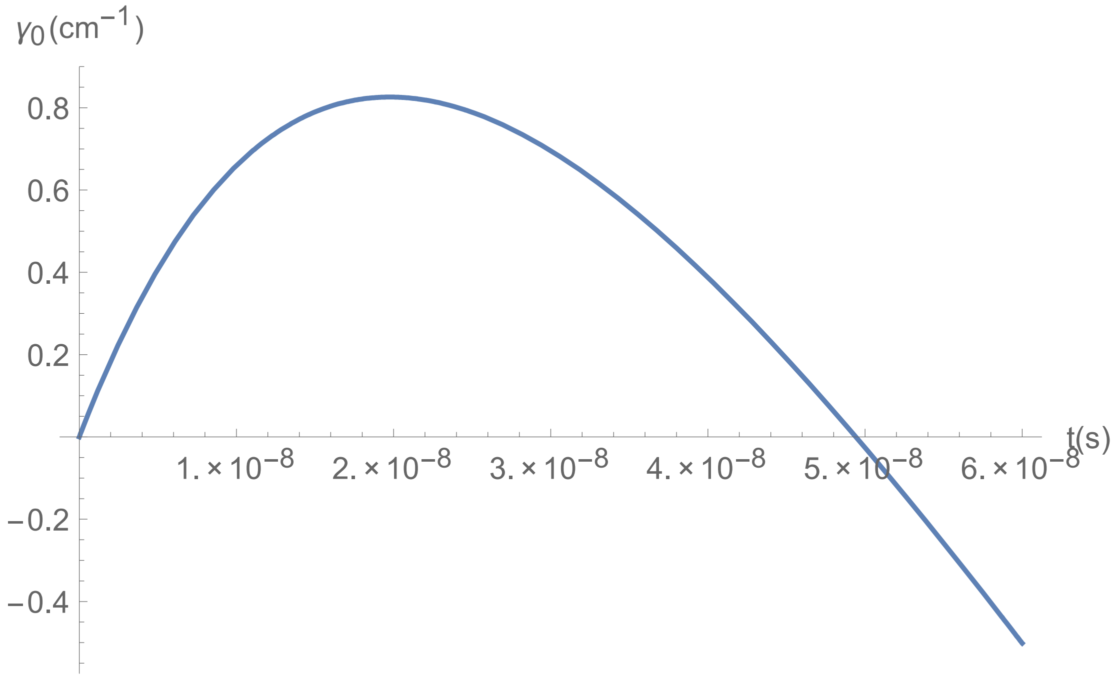
As we can see from the above figure, only when $t < 2 \times 10^{-7}s$, there is population inversion $N_2 > N_1$. For a CW operation laser, we want the system to be always in the population inversion state, which is not the case here.

To plot the small gain coefficient, we can use (7.5.2).

$$\gamma_0 = \sigma \times (N_2 - \frac{g_2}{g_1} N_1)$$

$$\sigma = 10^{12} cm^{-2}$$

From the figure for this problem, the degeneracy of each energy level are $g_1 = 2J_1 + 1 = 5$ and $g_2 = 2J_2 + 1 = 3$.



Problem 4.3 (Verdeyen Problem #8.12)

(a)

First, I started from (8.3.2a). It has been specified that we have negligible population in state 1. Hence, the rate equation is the following.

$$\frac{dN_2}{dt} = R_2(t) - \frac{N_2}{\tau_2} - \frac{\sigma(\nu)I_\nu}{h\nu}N_2$$

Assume we're in equilibrium state of the system. $\frac{dN_2}{dt} = 0$

$$N_2(t) = \frac{R_2(t)}{(\frac{1}{\tau_2} + \frac{\sigma(\nu)I_\nu}{h\nu})} = \frac{R_2(t)\tau_2}{(1 + \frac{\sigma(\nu)I_\nu}{h\nu}\tau_2)} = \frac{R_2(t)\tau_2}{(1 + \frac{I_\nu}{I_s})}$$

The spontaneous emission from the side should be always proportional to the population at state 2, meaning $P \propto N_2$. Assume the system is at constant pumping rate.

$$P = \frac{R_2(t)\tau_2}{(1 + \frac{I}{I_s})}$$

Because the one of the mirror is blocked, there is no cavity to confine the photons to generate stimulated emission, meaning $I_\nu = 0$.

$$P_0 = \frac{R_2(t)\tau_2}{(1 + \frac{I=0}{I_s})} = R_2(t)\tau_2$$

$$P = \frac{P_0}{(1 + \frac{I}{I_s})}$$

(b)

$$I = 100W/cm^2$$

$$\frac{P}{P_0} = 0.5$$

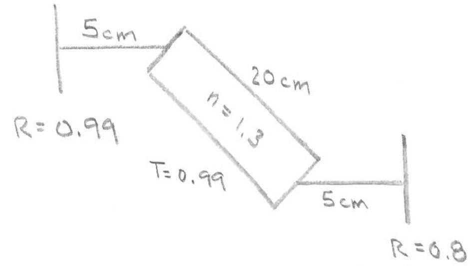
$$1 + \frac{I}{I_s} = 2$$

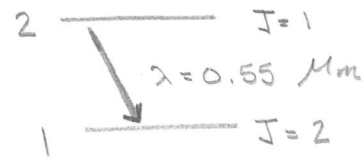
$$\frac{I}{I_s} = 1$$

$$I_s = 100W/cm^2$$

4.4 V# 8.16

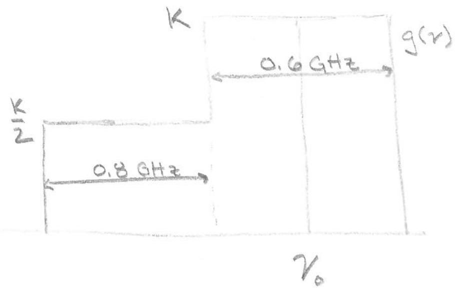
Stephanie Sanders
10/26/16





$A_{21} = 4 \times 10^5 \ s^{-1}$

A)



$$\int g(\nu) = 1 = K \cdot 0.6\,\text{GHz} + \frac{K}{2}(0.8\,\text{GHz})$$

$$1\,\text{GHz}\,(K) = 1 \rightarrow K = \frac{1}{1\times10^9 \frac{1}{s}} = 1\times10^{-9}\,s = 1\,ns$$

$$\boxed{g(\nu_0) = K = 1\,ns}$$

B) $$\sigma(\nu_0) = \frac{A_{21}\lambda^2}{8\pi n^2} g(\nu_0) = \frac{(4\times10^5\,s^{-1})(0.55\times10^{-6}\,m)^2}{8\pi(1.3)^2}(1\times10^{-9}\,s)$$

$$\boxed{\sigma(\nu_0) = 2.85\times10^{-18}\,m^2}$$

C) $$\tau_p = \left(\frac{2d_1}{c} + \frac{2nd_2}{c}\right)\left(\frac{1}{1-R_1R_2T^4}\right) = \left(\frac{2(0.1\,m)}{3\times10^8\,m/s} + \frac{2(1.3)(0.2m)}{3\times10^8\,m/s}\right)\left(\frac{1}{1-(0.99)(0.8)(0.99)^4}\right)$$

$$\boxed{\tau_p = 1.00\times10^{-8}\,s = 10.0\,ns}$$

D) Orientation of optical electrical field?

Brewsters angle requires p polarized light

p polarized light → electric field along x.

E) $$N_{th} = \frac{1}{2\sigma l_g} \ln\left(\frac{1}{R_1 R_2 T^4}\right)$$

$$= \frac{1}{2(2.85\times10^{-18}\,m^2)(0.2\,m)} \ln\left(\frac{1}{(0.99)(0.8)(0.99)^4}\right)$$

$$= 2.40\times10^{17}\ m^{-3}$$

$$N_2 - \frac{g_2}{g_1} N_1 = N_{th}$$

$$N_2 = 2.40\times10^{17}\ m^{-3} + \frac{3}{5}\,10^{18}\ m^{-3}$$

$$\boxed{N_2 = 8.40\times10^{17}\ m^{-3}}$$

Problem 4.4: V#8.16

```
In[1]:= A21 = 4*^5;
        λ = .55*^-6;
        n = 1.3;
        g = 1*^-9;
```

B)

```
In[5]:= σ = A21 * λ^2 / (8 Pi * n^2) * g
```

Out[5]= 2.84878×10^{-18}

C)

```
In[6]:= d1 = .1;
        d2 = .2;
        c = 3*^8;
        R1 = .99;
        R2 = .8;
        T = .99;
```

```
In[12]:= tp = (2 d1 / c + 2 * n * d2 / c) (1 / (1 - R1 * R2 * T^4))
```

Out[12]= 1.00331×10^{-8}

E)

```
In[13]:= Nth = 1 / (2 * σ * d2) Log[1 / (R1 * R2 * T^4)]
```

Out[13]= 2.39923×10^{17}

```
In[14]:= N1 = 10^(18);
```

```
In[15]:= N2 = Nth + 3 / 5 N1
```

Out[15]= 8.39923×10^{17}

E)

```
In[13]:= v0b = 1 / (λ * 10^2) // N
Out[13]= 15802.8

In[14]:= us = 166658.484;
         ls = us - v0b
Out[15]= 150856.

In[16]:= us / 8065.5
Out[16]= 20.6631

In[17]:= ls / 8065.5
Out[17]= 18.7038
```

F)

```
In[18]:= η = (us - ls) / us
Out[18]= 0.0948213
```

G)

```
In[19]:= g = Sqrt[4 * Log[2] / Pi] / dvd
Out[19]= 5.17252 × 10^-10

In[20]:= A21 = 6.56*^6;

In[21]:= σ = A21 * λ^2 / (8 Pi) * g
Out[21]= 5.40629 × 10^-17
```

H)

```
In[22]:= N1 = 10^16;
         g1 = 5;
         g2 = 3;

In[25]:= N2 = 0.05 / σ + g2 / g1 * N1
Out[25]= 6.92485 × 10^15
```

Problem 4.5 (Verdeyen Problem #8.20)

From (8.7.3a), we have output intensity of ASE.

$$I^{+}(\nu, l_g) = \frac{8\pi n^2 h\nu^3}{c^2} \frac{N_2}{N_2 - \frac{g_2}{g_1} N_1} [G_0(\nu) - 1] \frac{d\Omega}{4\pi}$$

To determine the FWHM, first we focus on the $[G_0(\nu) - 1]$ term. The reason I ignore $\frac{8\pi n^2 h\nu^3}{c^2}$ is because it's a monotonically increasing term with frequency.

$$G_0(\nu) = e^{\gamma_0(\nu) l_g}$$

$$G_0(\nu_0) = e^{\gamma_0(\nu_0) l_g}$$

$$\gamma_0(\nu) l_g = \gamma_0(\nu_0) \frac{(\Delta\nu_h/2)^2}{(\nu - \nu_0)^2 + (\Delta\nu_h/2)^2} \times l_g$$

FWHM is defined as the intensity drops to half of the maximum value.

$$[G_0(\nu) - 1] = 1/2 \times [G_0(\nu_0) - 1]$$

Change variable for the above equations.

Let

$$\Delta\nu_H = \frac{\nu - \nu_0}{\Delta\nu_h/2}$$

$$\gamma_0(\nu) l_g = \gamma_0(\nu_0) \frac{1}{(\Delta\nu_H)^2 + 1} \times l_g$$

$$e^{\frac{\gamma_0(\nu_0) \times l_g}{(\Delta\nu_H)^2 + 1}} - 1 = 1/2 \times [G_0(\nu_0) - 1]$$

$$e^{\frac{\gamma_0(\nu_0) \times l_g}{(\Delta\nu_H)^2 + 1}} - 1 = 0.5(e^{\gamma_0(\nu_0) l_g} - 1)$$

$$e^{\frac{\gamma_0(\nu_0) \times l_g}{(\Delta\nu_H)^2 + 1}} - 1 = 0.5(G_0(\nu_0) - 1)$$

$$e^{\frac{\gamma_0(\nu_0) \times l_g}{(\Delta\nu_H)^2 + 1}} = 0.5(G_0(\nu_0) + 1)$$

$$\frac{\ln(G_0(\nu_0))}{(\Delta\nu_H)^2 + 1} = \ln(0.5(G_0(\nu_0) + 1))$$

$$(\Delta\nu_H)^2 = \frac{\ln(G_0(\nu_0))}{\ln(0.5(G_0(\nu_0) + 1))} - 1$$

To switch gain to dB unit, we need to change the gain variable.

Let's assume a new variable in dB unit, "t".

$$t(dB) = 10 \times Log(G_0(\nu_0))$$

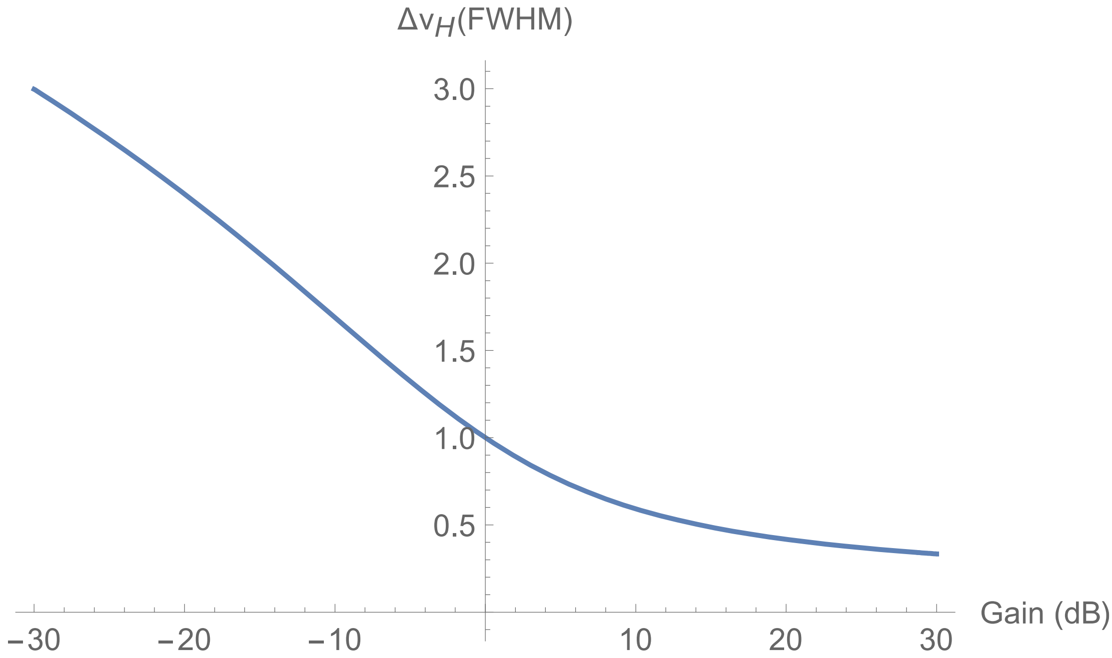
$$0.1t = Log(G_0(\nu_0))$$

$$10^{0.1t} = G_0(\nu_0)$$

So we substitute this new variable into the equation.

$$\Delta\nu_H = \sqrt{\frac{\ln(10^{0.1t})}{\ln(0.5(10^{0.1t}+1))} - 1}$$

Plot this function with Mathematica.



Problem 4.6 (Verdeyen Problem #21)

(a)

From (6.4.2b), we have the photon lifetime in passive cavity.

$$\tau_p = \frac{2nd/c}{1 - R_1 R_2}$$

Here we need to add the additional loss from lens and gain medium.

$$\tau_p = \frac{3d/c}{1 - 0.98 \times 0.98 \times 0.9 \times (0.98)^2 \times (0.97)^2} = 22.838nsec$$

(b)

From (6.4.4), we can calculate the quality factor.

$$Q = \omega_0 \tau_p$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi c}{\lambda_0} = 2.356 \times 10^{15}\, rad/s$$

$$Q = 5.381 \times 10^7$$

(c)

$$FSR = \frac{c}{3nd} = 2 \times 10^8 Hz$$

(d)

Minimum gain has to at least overcome the loss in the cavity.

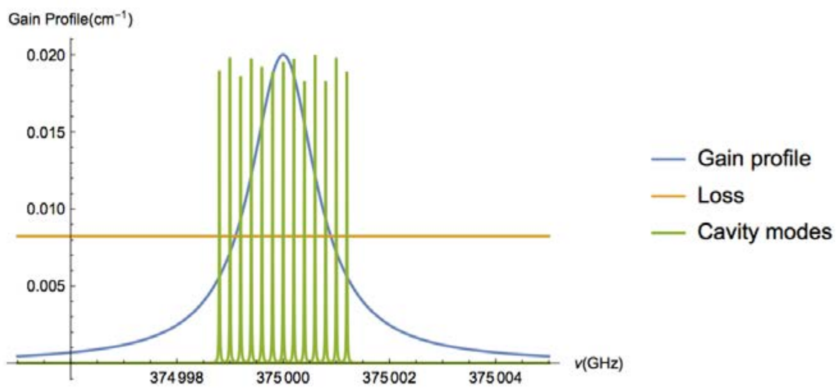
$$e^{\gamma_0(\nu_0)\times l_g} \times R_1 R_2 R_3 T_1 T_2 (0.97)^2 > 1$$

$$\gamma_0(\nu_0) > \frac{-\ln(R_1 R_2 R_3 T_1 T_2 (0.97)^2)}{l_g}$$

$$\gamma_0(\nu_0) > 0.0082cm^{-1}$$

(e)

We have the FSR and optical frequency from above calculations. Now I plot the cavity mode, gain spectrum, and loss level. We can see how many modes are above the loss. There are **9 modes** can overcome the loss, which are above threshold.



(f)

From (7.4.7), simulated emission cross section is defined as following.

$$\sigma(\nu_0) = A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu_0)$$

$$\sigma(\nu_0) = 1.08 \times 10^{-14} cm^2$$

(g)

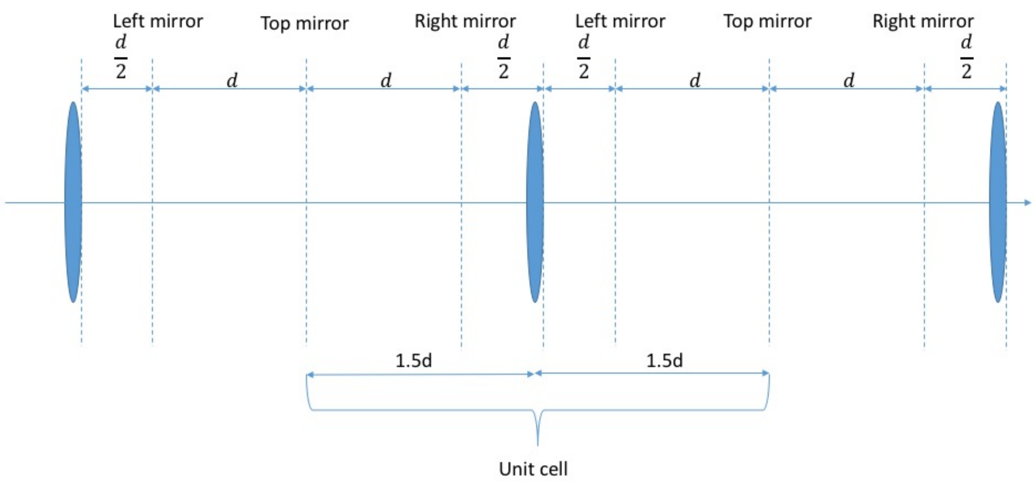
From (7.5.4), we have the absorption cross section definition.

$$\sigma_{abs}(\nu) = (g2/g1)\sigma_{stim}(\nu)$$

$$\sigma_{abs}(\nu_0) = \frac{3}{5}\sigma_{stim}(\nu_0) = 6.48 \times 10^{-15} cm^2$$

(h)

(1) This problem is very similar to Problem 5.1 in Verdeyen. Let's draw the equivalent lens waveguide.



By symmetry, the minimum beam waist, z=0, should appear in the middle of the two lenses, which is exactly the position of the top mirror.

(2)

Similar to (3.5.1), we need to find the phase match condition for this cavity.

$$k \times 3d/2 - (1+m+p)tan^{-1}\left(\frac{3d}{2z_0}\right) = q\pi$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$$

$$\frac{2\pi\nu}{c} \times 3d - 2(1+m+p)tan^{-1}\left(\frac{3d}{2z_0}\right) = 2q\pi$$

$$\nu_{m,p,q} = \frac{c}{6\pi d}(2q\pi + 2(1+m+p)tan^{-1}\left(\frac{3d}{2z_0}\right))$$

(3)

$$z_0 = 0.58m$$

$$\nu_{1,0,q} - \nu_{0,0,q} = \frac{c}{6\pi d}2tan^{-1}\left(\frac{3d}{2z_0}\right) = 58MHz$$

Problem 4.7 (Verdeyen Problem #8.35)

(a)

$$\frac{N_2^e}{N_0^e} = \frac{1}{8}$$

$$\frac{N_1^e}{N_0^e} = \frac{1}{4}$$

$$N_0^e + N_1^e + N_2^e = N = 2.2 \times 10^{20} cm^{-3}$$

$$\frac{11}{8} N_0^e = 2.2 \times 10^{20} cm^{-3}$$

$$N_0^e = 1.6 \times 10^{20} cm^{-3}$$

$$N_1^e = 4 \times 10^{19} cm^{-3}$$

$$N_2^e = 2 \times 10^{19} cm^{-3}$$

(b) Skip because there is no absorption cross section.

(c)

(1)

We can modify (8.3.2ab) to get the rate equations.

There is no pumping to state 1, so $R_1 = 0$. Neglect the stimulated emission.

$$\frac{dN_2}{dt} = R_2 - \frac{N_2 - N_2^e}{\tau_2}$$

$$\frac{dN_1}{dt} = \frac{N_2 - N_2^e}{\tau_2} - \frac{N_1 - N_1^e}{\tau_1}$$

(2)

Assume the system is in steady state, meaning all derivatives should be equal to zero.

$$\frac{N_2 - N_2^e}{\tau_2} = R_2$$

$$\frac{N_2 - N_2^e}{\tau_2} = \frac{N_1 - N_1^e}{\tau_1}$$

For optical transparency, it means the populations in both state 1 and 2 should be equal. So there is no absorption.

$$N_1 - N_1^e = R_2 \tau_1$$

$$N_2 - N_2^e = R_2 \tau_2$$

$$N_2 = N_1$$

$$N_2 = R_2 \tau_2 + N_2^e$$

$$R_2 \tau_2 + N_2^e - N_1^e = R_2 \tau_1$$

$$R_2 = \frac{N_2^e - N_1^e}{\tau_1 - \tau_2}$$

(3)

$$R_2 = \frac{1}{\tau_1} \frac{N_2^e - N_1^e}{1 - \tau_2/\tau_1}$$

From (a), we know the difference between equilibrium concentrations of state 2 and 1 are less than zero.

$$N_2^e - N_1^e < 0$$

It's obvious that we need $\tau_2/\tau_1 > 1$ to have a positive pumping rate for optical transparency. Absorption coefficient should be proportional to $N_1 - N_2$. Because it need more populations in the lower state in order to have absorption.

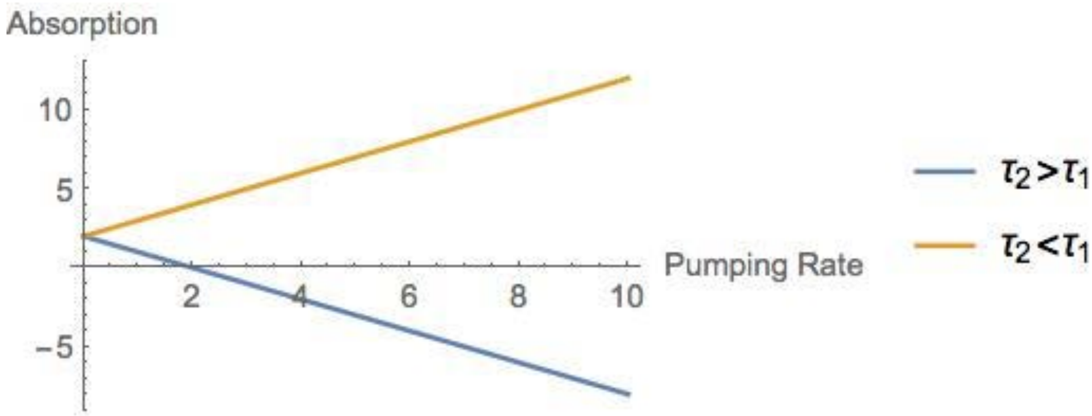
$$\alpha = \sigma(N_1 - N_2)$$

$$\alpha = \sigma(N_1^e - N_2^e + R_2(\tau_1 - \tau_2))$$

For simplicity, I am using following parameters for the plot.

$$N_1^e = 4$$

$$N_2^e = 2$$



Problem 4.8 (Verdeyen Problem #9.1)

(a)

From (7.6.17), we have the Doppler broadening linewidth.

$$\Delta\nu_D = \sqrt{\frac{8kTln2}{Mc^2}}\nu_0$$

$$T = 273 + 300 = 573K$$

$$M = 20amu$$

$$\nu_0 = \frac{c}{6328\text{Å}} = 4.74 \times 10^{14}Hz$$

$$\Delta\nu_D = 1.81GHz$$

(b)

From (7.2.4b), we have the number of modes per unit volume in blackbody radiation.

$$p(v)dv = \frac{8\pi n^3 v^2}{c^3}dv$$

$$N = \frac{8\pi n^3 v^2 \Delta\nu_D}{c^3}V = 3.799 \times 10^8$$

(c)

$$FSR = \frac{c}{2nd} = 150MHz$$

$$\frac{\Delta\nu_D}{2} = 905MHz$$

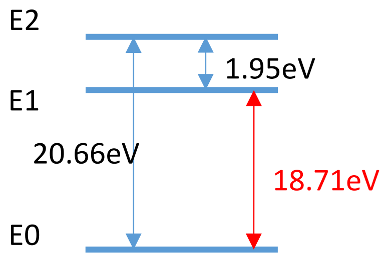
There are 13 modes within $\pm\frac{\Delta\nu_D}{2}$.

(d)

Combine the two previous results. We can calculate the probability.

$$p = \frac{13}{3.799 \times 10^8} = 3.42 \times 10^{-8}$$

(e) and (f)



$$Transition\ energy = \frac{1.24}{0.6328} = 1.95954eV$$

$$Upper\ state\ energy = 20.6657eV$$

$$\eta = \frac{1.95}{20.66} = 9.44\%$$

(g)

$$\sigma(\nu_0) = A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu_0) = 3.676 \times 10^{-13} cm^2$$

$$g(\nu_0) = \sqrt{\frac{4ln2}{\pi}}\frac{1}{\Delta\nu_D} = 5.19 \times 10^{-10}$$

$$\sigma(\nu_0) = 5.42 \times 10^{-13} cm^2$$

(h)

$$\gamma_0 = \sigma(\nu_0)\left(N_2 - \frac{g_2}{g_1}N_1\right) = 0.05m^{-1}$$

$$g_2 = 2J_2 + 1 = 3$$

$$g_1 = 2J_1 + 1 = 5$$

$$N_1 = 10^{10} cm^{-3}$$

$$N_2 = 6.92 \times 10^9 cm^{-3}$$