ECE 4300: Lasers and Optoelectronics
Fall 2016, Debdeep Jena and Clif Pollock, Cornell University
Solutions to Assignment 5: By Kevin Lee, Athith Krishna and Reet Chaudhuri

Problem: 5.1

An Ar-ion laser oscillating on its green λ=514.5 nm transition has a total loss per pass γ=4%, an unsaturated peak gain $G_p=\exp(\sigma_p N l)=1.3$, and a cavity length L=100 cm. To select a single-longitudinal mode, a tilted and coated quartz (n_r=1.45) Fabry-Perot etalon with a l=2 cm tickness is used inside the resonator. Assuming for simplicity that one cavity mode is coincident with the peak of the transition (whose linewidth is $\Delta\nu_0$=3.5 GHz), calculate the etalon finesse and the reflectivity of the two etalon faces to ensure single-mode operation.

7.20A Single longitudinal mode selection by an intracavity etalon.

Assuming for the sake of simplicity that both the cavity length and the etalon tilting angle are tuned so that one cavity resonance and one transmission peak of the etalon coincide with the center of the gain line, the net gain per-transit experienced by the resonant mode is given by $G_p \exp(-\gamma)\cong 1.25$, where G_p=1.3 is the unsaturated peak gain and γ=0.04 are the total losses per-pass, including output coupling loss and internal losses. If $\Delta\nu=c/2L\cong 150$ MHz is the frequency separation between adjacent longitudinal modes of the laser cavity, single longitudinal mode operation is ensured provided that all off-resonance longitudinal modes experience a loss larger than the gain, i.e.:

$$\exp(-\gamma)G_p g^*(m\Delta\nu)T(m\Delta\nu) < 1 \qquad (1)$$

for any mode $m=\pm1,\pm2,\pm3,\ldots$. In Eq.(1), $g^*(\nu)=\exp[-\ln 2(\nu/\Delta\nu_0)^2]$ is the Gaussian curve of the Doppler-broadened gain transition, $\Delta\nu_0$=3.5 GHz is the FWHM of the curve, and $T(\nu)$ is the transmission function of the intracavity etalon. $T(\nu)$ is given by [see Eq.(4.5.6) of PL]:

$$T(\nu) = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\left[\dfrac{2\pi l'\nu}{c}\right]} \qquad (2)$$

where R is the reflectivity of the two etalon faces and $l'=n_r l\cos\theta \cong n_r l=2.9$ cm for a small tilting angle θ. Notice that the free-spectral-range of the etalon, given by $\Delta\nu_{fsr}=c/2l' \cong 5.17$ GHz, is much larger than the cavity axial mode separation $\Delta\nu$ and comparable to the width of the gain curve. For longitudinal modes that fall under the first lateral peaks of the etalon transmission function, it is straightforward to show that Eq.(1) is satisfied since $G_p\exp(-\gamma)g^*(\Delta\nu_{fsr})\cong 0.27<1$. Thus in order to ensure single longitudinal mode operation it is sufficient that Eq.(1) be satisfied in correspondence of the two longitudinal cavity modes adjacent the resonant mode, i.e. for $m=\pm 1$. This condition yields:

$$T(\Delta\nu) < \frac{1}{G_p \exp(-\gamma) g^*(\Delta\nu)} \cong 0.8 \qquad (3)$$

which, with the help of Eq.(2), can be written as:

$$(1-R)^2 < 0.8\left[(1-R)^2 + 4R\,sin^2\left(\pi\frac{\Delta\nu}{\Delta\nu_{fsr}}\right)\right] \qquad (4)$$

Since $\varepsilon=\sin^2(\pi\Delta\nu/\Delta\nu_{fsr})\cong 8.3\times 10^{-3}$ is a small quantity, the inequality (4) can be easily solved with respect to (1-R), yielding at leading order $1-R< 4\varepsilon^{1/2}$, i.e. $R>64\%$. A more accurate estimation of R, obtained by an iterative procedure, yields $R>70\%$. The minimum finesse of the etalon is thus given by:

$$F = \frac{\pi R^{1/2}}{1-R} \cong 0.87 \qquad (5)$$

Problem 5.2 (Verdeyen Problem # 8.32)

(a) and (b)

First we can calculate the loss in the system, α.

Without pumping ($\gamma_0 = 0$), the transmission is 0.85.

$$T = 0.85 = e^{-\alpha l_g}$$

$$\alpha = \frac{-1}{l_g}\ln(0.85) = 3.25 \times 10^{-4} cm^{-1}$$

For small signal gain, it means the optical intensity is much smaller than the saturation intensity.

$$I \ll I_s$$

In this case, we can use (8.3.18).

$$G_0 = \frac{I_{out}}{I_{in}} = e^{l_g(\gamma_0 - \alpha)} = 6.2807 = 7.98\text{dB}$$

(c)

For a unity gain amplifier, it means that $I_{in} = I_{out}$. In this case, the derivative of the intensity along z-axis is zero.

$$\frac{dI}{dz} = 0$$

$$\frac{\gamma_0}{1 + I/I_s} - \alpha = 0$$

$$\frac{\gamma_0}{\alpha} = 1 + \frac{I}{I_s}$$

$$I = \left(\frac{\gamma_0}{\alpha} - 1\right) I_s = 180.923\text{W}/cm^2$$

Problem 5.3 (An Optical Amplifier)

Verdeyen Problem # 9.6.

9.6. An optical amplifier of length $l_g = 10$ cm uses a homogeneously broadened transition centered at $\lambda_0 = 760$ nm, with a stimulated emission cross section of 2×10^{-20} cm^2, an upper-state lifetime of 1.54 ms, and a negligible lower-state lifetime. There is some residual loss of $\alpha = 0.01$ cm^{-1}, which is independent of intensity whereas the gain coefficient saturates according to the homogeneous law.

(a) Find the inversion necessary to obtain a net small-signal amplification of 6 dB.

(b) What is the value of the saturation intensity (in W/cm^2)?

(c) At what value of the input intensity will the net gain of this amplifier be 3 dB (i.e., $I_{out}/I_{in} = 2$)?

Solution:

(a) Given

$$G_0 = 6\,dB = 4 = \exp\left((\gamma_0 - \alpha)l_g\right)$$

Taking natural log on both sides,

$$(\gamma_0 - \alpha)l_g = 1.386 \Rightarrow \gamma_0 = \frac{1.386}{l_g} - \alpha = \frac{1.386}{10cm} - 0.01\,cm^{-1} = 0.1486\,cm^{-1}$$

We know,

$$\gamma_0 = \sigma\,\underbrace{(N_2 - \frac{g_2}{g_1}N_1)}_{inversion=\Delta N} \Rightarrow \Delta N = \frac{\gamma_0}{\sigma} = \frac{0.1486cm^{-1}}{2 \times 10^{-20}cm^2} = \boxed{7.43 \times 10^{18}cm^{-3}}$$

(b)

$$I_s = \frac{h\nu}{\sigma\tau_2} = \frac{hc}{760\text{ nm} \times 2 \times 10^{-20}cm^2 \times 1.54\text{ ms}} = \boxed{8.50\,\frac{kW}{cm^2}}$$

(c) Using homogenous law,

$$\left(\frac{1}{I}\right)\left(\frac{dI}{dz}\right) = \frac{\gamma_0 I}{1 + \frac{I}{I_s}} - \alpha \quad --(1)$$

Let, $x = \frac{I}{I_s}$, then (1) can be re-written in terms x as,

$$\frac{dx}{dz} = \frac{\left[x - \left(\left(\frac{\gamma_0}{\alpha}\right) - 1\right)\right]}{1}(-\alpha) = \frac{\left[x - \left(\left(\frac{\gamma_0}{\alpha}\right) - 1\right)\right]}{x}(-\alpha x) \quad --(2)$$

To solve (2), we use method of separation of variables, then partial fractions, rewrite (2) as,

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$$dz = dx\left[\frac{1}{\left[x - \left(\left(\frac{\gamma_0}{\alpha}\right) - 1\right)\right](-\alpha)}\right] \Rightarrow dz = \frac{1 + x}{\left[x^2 - \left(\left(\frac{\gamma}{\alpha}\right) - 1\right)x\right]}$$

$$-\alpha\, dz = \left[\frac{X_1}{x} + \frac{X_2}{x - \left(\left(\frac{\gamma_0}{\alpha}\right) - 1\right)}\right]dx \quad --- (3)$$

Let $X_1 = -\frac{1}{\left(\frac{\gamma}{\alpha}\right)-1}$; $X_2 = -X_1$; and $\chi = \frac{\gamma_0}{\alpha} - 1 = 1.386$

Integrating (3) with the limits,

$$-(\gamma_0 - \alpha)\int_0^{l_g} dz = \int_{x_1}^{x_2} \frac{dx}{x - \chi} - \int_{x_1}^{x_2} \frac{dx}{x}$$

$$-(\gamma_0 - \alpha)l_g = \ln\left[\frac{x_2 - \chi}{x_1 - \chi}\right] - \ln\left[\frac{x_2}{x_1}\right] \quad --- (4)$$

Given $\frac{I_{out}}{I_{in}} = 2$, therefore,

$$x_2 = 2x_1 = \frac{I_2}{I_s} = 2\left(\frac{I_1}{I_s}\right)$$

Taking exponential on both sides of (4),

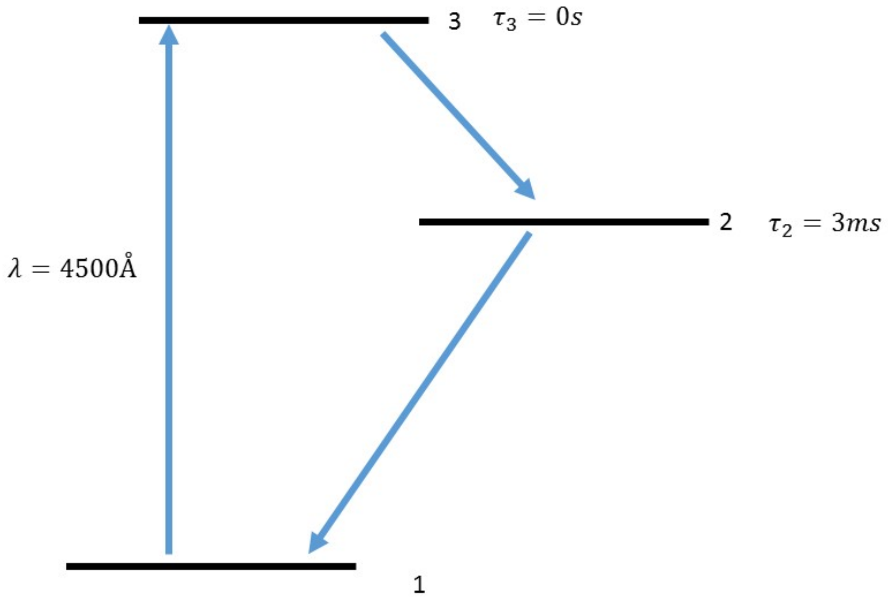
$$\exp\left[-(\gamma_0 - \alpha)l_g\right] = 0.5\left[\frac{2x_1 - \chi}{x_1 - \chi}\right] = 0.25 \Rightarrow x_1 = 4.62$$

$$x_1 = \frac{I_1}{I_s} \Rightarrow I_1 = \boxed{39.2\,\frac{kW}{cm^2}}$$

Problem 5.4 (Verdeyen Problem # 9.17)

(a)

$$N_1 + N_2 + N_3 = N = 1.58 \times 10^{19} atoms/cm^3$$



Since there is no population in state 3, its lifetime is zero.

$$N_1 + N_2 = N$$

To calculate the required power, we need to know the energy from state 1 to 3.

$$E_{g,31} = \frac{hc}{\lambda} = 4.417 \times 10^{-19} J$$

This is the energy required for pumping one electron from state 1 to 3.

$$N_2 = 10^{19} cm^{-3}$$

For a steady state condition, the population of state 2 should be the product of pumping rate and lifetime of state 2.

$$N_2 = R_{31} \times \tau_2$$

$$R_{31} = 3.33 \times 10^{21} s^{-1} cm^{-3}$$

$$V(Gain\ medium\ volume) = 1cm^2 \times 15cm = 15cm^3$$

$$\text{Power} = R_{31} \times E_{g,31} \times V = 22086W$$

(b)

Spontaneous emission power can be easily calculated by following equation.

$$Spontaneous\ emission\ power = \frac{N_2}{\tau_2} \times h\nu_{21} \times V$$

$$h\nu_{21}: Photon\ energy\ from\ state\ 2\ to\ 1$$

Because this problem doesn't specify the transition wavelength from state 2 to 1. I am going to use the emission wavelength specified on page 293, which is 6943Å. Put in all numbers into the equation.

$$Spontaneous\ emission\ power = 14315.1W$$

(c)

To calculate the maximum output power, we can utilize (9.4.13a).

$$P(\max) = \eta_{cpl}\frac{h\nu N_p(\max)}{\tau_p}$$

First, let's look for passive photon lifetime.

From (9.4.2c), we have the photon lifetime.

$$\tau_p = \frac{\tau_{RT}}{1-S}$$

$$\tau_{RT} = \frac{2l_g}{c/n_g} + \frac{2(d-l_g)}{c} = 2.1nsec$$

$$\tau_p = \frac{\tau_{RT}}{1-R_1R_2} = 6.3nsec$$

According to the 2nd equation in page 291, we can calculate the coupling efficiency.

$$\eta_{cpl} = \frac{T_2}{1-S} = \frac{0.3}{1-0.95\times 0.7} = 0.89522$$

$$N_p(\max) = \frac{n_i - n_{th}}{2} - \frac{n_{th}}{2}\ln(\frac{n_i}{n_{th}})$$

From (9.4.16a), we can calculate the initial atoms.

$$n_i = [N_{2i} - N_{1i}]Al_g = [10^{19} - 0.58\times 10^{19}]\times 15\times 1 = 6.3\times 10^{19}atoms$$

To achieve threshold condition, we need to overcome the internal loss.

$$\gamma_{th} = \frac{1}{2l_g}\ln\left(\frac{1}{R_1R_2}\right) = \sigma(N_2 - N_1)_{th}$$

$$(N_2 - N_1)_{th} = \frac{1}{2l_g\sigma}\ln\left(\frac{1}{R_1R_2}\right) = 1.07\times 10^{18}cm^{-3}$$

$$n_{th} = (N_2 - N_1)_{th}Al_g = 1.6\times 10^{19}atoms$$

$$N_p(\max) = 1.25\times 10^{19}atoms$$

$$P(\max) = 509.989MW$$

(d)

From (9.4.16c), we can calculate the output pulse energy.

$$W_{out} = \eta_{xtn}\eta_{cpl}\frac{n_i h\nu}{2}$$

Extraction efficiency can only be solved analytically from (9.4.14).

$$\frac{n_i}{n_{th}} = \frac{6.3\times 10^{19}}{1.6\times 10^{19}} = 3.9375$$

From FIGURE 9.13, extraction efficiency is around 0.98.

$$W_{out} = 7.9121\, Joul$$

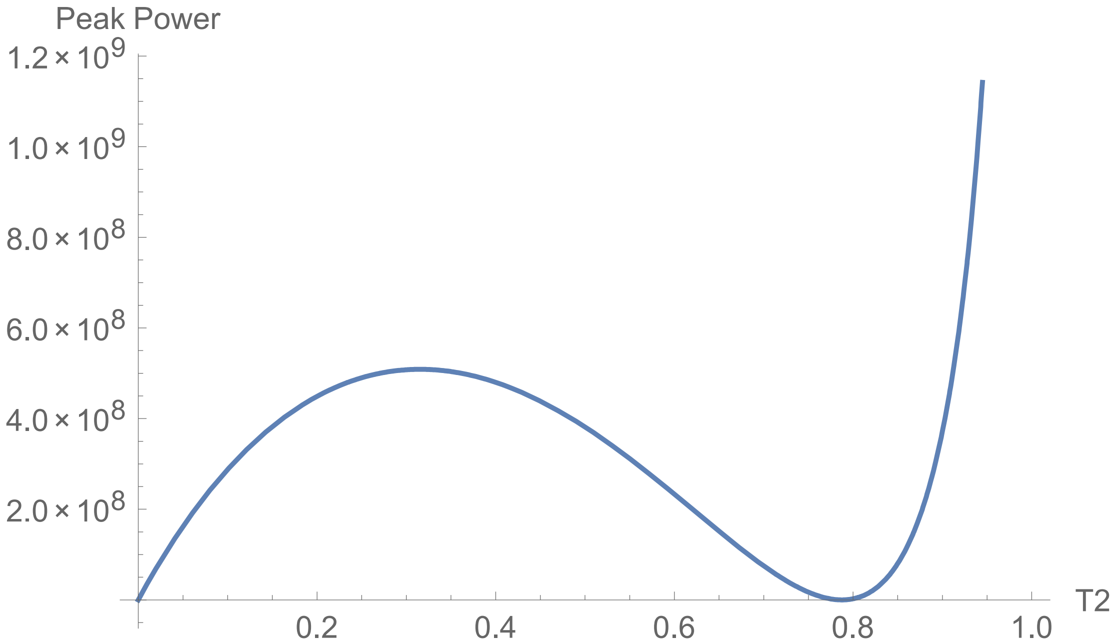
(e)

$$\Delta t \approx \frac{W_{out}}{P_0(\max)} = \frac{7.9121J}{509.989MW} = 15.51nsec$$

From FIGURE 9.14, the approximate linewidth is 2.5.

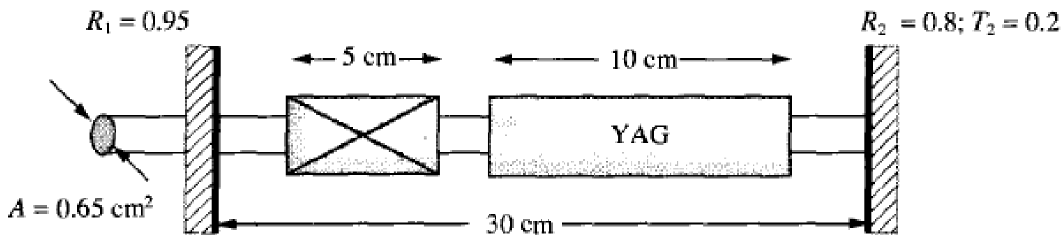
$$\Delta t = 2.5 \times \tau_p = 15.75nsec$$

(f)



Problem 5.5 (A Q-switched YAG Laser)
Verdeyen Problem # 10.25.

10.25. The YAG laser shown below is pumped to three times threshold before the electro-optic shutter is opened for Q switching. Assume (1) the characteristics of the YAG rod are those specified in Tables 10.2 and 10.3 but ignore the scattering loss, (2) the transmission at all air-device interfaces is 0.99, (3) the index of refraction of the electro-optic switch is 2.3, (4) lasing at 1.0615 μm (where there is no significant overlap with any other transition), (5) the populations in the $^4F_{3/2}$ states are always distributed according to the Boltzmann relation with $kT = 208$ cm^{-1}, even within the Q-switched pulse.



(a) Evaluate the following parameters to be used in the calculation: photon lifetime of passive cavity in ns, A_{21} coefficient for the transition in sec^{-1}, stimulated emission cross section in cm^2, initial density of atoms in $^4F_{3/2}$ manifold in cm^{-3}, and final density of atoms in $^4F_{3/2}$ manifold in cm^{-3}.

(b) Compute the peak power in watts, the energy in joules, and determine the pulse width using the theory of Sec. 9.4.

(c) The theory of Sec. 9.4 is not quite applicable to this problem since the lifetime of the lower state is only 30 ns and the time scale for the establishment of a Boltzmann factor among the levels of the $^4I_{11/2}$ manifold is even shorter. Redo the calculations of (a) assuming that $N_1 = 0$, which is a bit different from the analysis of Sec. 9.4.

Solution:

(a)

$$\tau_p = \frac{\tau_{RT}}{1-S} = \frac{\left[\frac{2}{c}\cdot\left(n_1(15cm) + n_2(5cm) + n_3(10cm)\right)\right]}{[1 - R_1R_2T^8]}$$

$$= \frac{\left[\frac{2}{c}\cdot\left(\{1\}(15cm) + \{2.3\}(5cm) + \{1.816\}(10cm)\right)\right]}{[1-(0.95\cdot 0.8\cdot(0.99)^8)]} = 9.97\times 10^{-9}s$$

$$\boxed{= 9.97\ ns}$$

At threshold,

$$\gamma_{th} = \frac{1}{2l_g}\ln\frac{1}{S} = \frac{1}{2l_g}\ln\frac{1}{R_1R_2T^8} = \frac{1}{2(10)}\ln\frac{1}{0.95\cdot 0.8\cdot (0.99)^8} = 1.78\times 10^{-2}cm^{-1}$$

Using values from the table,

$$A_{21} = \frac{Branching\ Ratio}{Fluoroscent\ lifetime\ of\ \ ^4F_{\frac{3}{2}}} = \frac{0.0799}{255\mu s} = \boxed{313.3\ s^{-1}}$$

$$\sigma = \frac{A_{21}\lambda^2}{8\pi n^2}\left(\frac{2}{\pi\Delta\nu}\right) = \frac{313.3\ s^{-1}(\lambda^2)}{8\pi n^2}\left(\frac{2}{\pi(108\times 10^9\text{Hz})}\right) = \boxed{2.51\times 10^{-19}cm^2}$$

It is given that the transition between the levels is from $11{,}423\ cm^{-1}$ to $2002\ cm^{-1}$,

It is also given that, $\frac{n_i}{n_{th}} = 3$

Initially,

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right)$$

$$\frac{N_3}{N_1} = \exp\left(-\frac{E_3 - E_1}{kT}\right)$$

$$[N]_{lower} = 7.09\times 10^{16}cm^{-3}$$

$$[N]_{upper} = 4.73\times 10^{16}cm^{-3}$$

Using Eq. 9.4.14,

$$[N]_{th} = 3.53\times 10^{17}cm^{-3}$$

From this, we get n_{th}, n_i

$$n_{th} = 7.67\times 10^{17}\ \#$$

$$n_i = 3n_{th} = 2.3\times 10^{18}\ \#$$

As we know $\frac{n_i}{n_{th}} = 3$, we can use Fig 9.13 in Verdeyen to get the $\eta_{xtn} = \frac{n_i - n_f}{n_i} = 0.941$

$$\eta_{cpl} = \frac{T_2}{1-S} = \frac{0.2}{1-R_1R_2T^8} = \frac{0.2}{1-0.95\cdot 0.8\cdot (0.99)^8} = 0.669$$

$$[N]_{final} = 2.1\times 10^{16}cm^{-3}$$

(b)

From part (a), we know η_{cpl} and η_{xtn},

$$W = \eta_{xtn}\eta_{cpl}\left(\frac{n_i h\nu}{2}\right) = (0.941)(0.669)\left(\frac{2.3\times 10^{18}\ h\nu}{2}\right) = \boxed{0.1353\ J}$$

Using Eq. 9.4.13a

$$P_{max} = \eta_{cpl} \cdot \frac{h\nu}{\tau_p}\left(N_{p_{max}}\right) = \eta_{cpl} \cdot \frac{h\nu}{\tau_p}\left[\frac{n_i - n_{th}}{2} - \left(\frac{n_{th}}{2}\right)\ln\left(\frac{n_i}{n_{th}}\right)\right]$$
$$= 0.669 \cdot \frac{h\nu}{\tau_p}\left[\frac{2.3 \times 10^{18} - 7.67 \times 10^{17}}{2} - \left(\frac{7.67 \times 10^{17}}{2}\right)\ln(3)\right]$$
$$\boxed{= 4.33 \times 10^6 W}$$

From Fig 9.14, and $\frac{n_i}{n_{th}} = 3$, we can calculate the pulse width

$$FWHM = \frac{\Delta t_{\frac{1}{2}}}{\tau_p} = 5.5 - 2.3 = 3.2 \Rightarrow \Delta t_{\frac{1}{2}} = (3.2)\tau_p = \boxed{31.9\ ns}$$

We can compare this to the result obtained from Eq. 9.4.17,

$$\Delta t_{\frac{1}{2}} = \frac{W}{P_{max}} = \frac{0.1353\ J}{4.33 \times 10^6 W} \approx \boxed{31.2\ ns}$$

Problem 5.6 (Saturable Absorber)

Verdeyen Problem # 9.23.

9.23. This problem predicts the results of an experiment in which we irradiate an *absorption* cell with an optical pulse approximated by

$$I(t) = I_0 \sin^2\left(\frac{\pi t}{T}\right) \qquad (T = 2\text{ ns})$$

The optical frequency is tuned to the center of the transition, which has an absorption cross section of 10^{-14} cm^2. The "small"-signal transmission through the cell is -30 dB. Graph the output intensity as a function of time for three values of energy contained in the pulse (a) $w = 2\,\mu\text{J/cm}^2$, (b) $w = 20\,\mu\text{J/cm}^2$, and (c) $w = 200\,\mu\text{J/cm}^2$. ($\lambda_0 = 5889$ Å).

Solution:

Given, $\lambda_0 = 588.9\ nm$ and $\sigma = 10^{-14}$

Using eq. 9.6.3b,

$$w_s = \frac{h\nu}{2\sigma} = \frac{hc}{2\lambda\sigma} = 16.86 \times 10^{-6}\frac{J}{cm^2}$$
$$G_0 = -30dB = 10^{-3}$$

Using Eq 9.6. _ (on P.315),

Problem 5.6 (Verdeyen Problem # 9.23)

To calculate the output intensity, we should use (9.6.11).

$$I_2(t) = I_1(t)\frac{G_0 e^{u_1(t)}}{1 + G_0[e^{u_1(t)} - 1]}$$

The small signal gain is already specified -30dB.

$$G_0 = 10^{-3}$$

Now we have to look for the pulse energy normalized by the characteristic saturation energy.

$$u_1(t) = \frac{w(t)}{w_s}$$

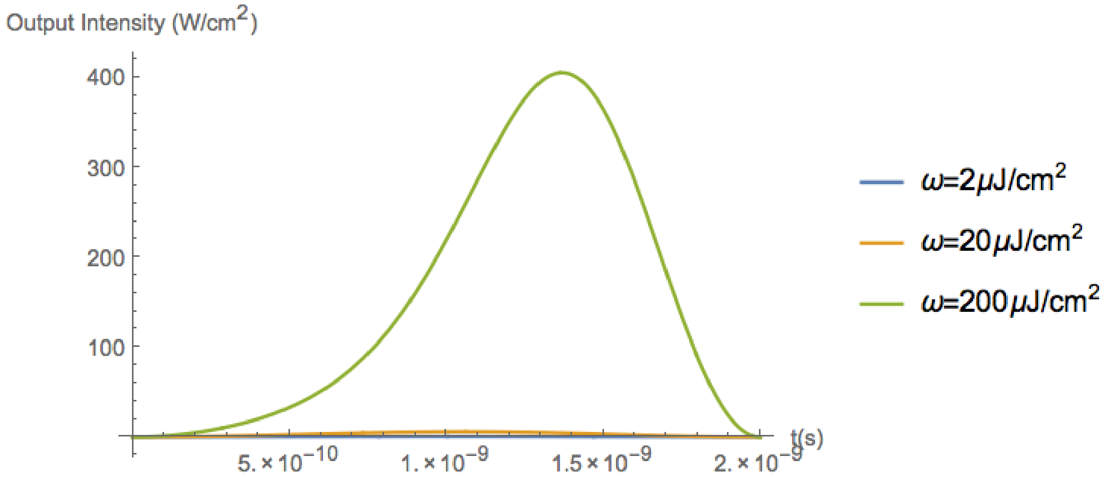
$$w(t) = \int_{-\infty}^{t} I_1(t)dt$$

$$w_s = \frac{h\nu}{\sigma} = 3.375 \times 10^{-5} J/cm^2 \quad = 33.75\mu J/cm^2$$

$$I_1(t) = \frac{w}{T} sin^2\left(\frac{\pi t}{T}\right) = \frac{w}{2T}[1 - \cos(\frac{2\pi t}{T})]$$

$$w(t) = \int_{-\infty}^{t} I_1(t)dt = \frac{w}{2T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right] = \frac{w}{2}[\frac{t}{T} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{T}\right)]$$

Plug in all the functions into the equation of output intensity.



Problem 5.7 (Verdeyen Problem # 9.27)

(a)

The power spectrum is given in the problem.

$$P(\omega) = P_0 \sum_{n=-\infty}^{n=\infty} sech^2\left(\frac{n\omega_c}{\Delta\omega}\right)\delta(\omega - [\omega_0 + n\omega_c])$$

To calculate the average power, we need to find instantaneous power. And it can be related to electric field.

$$P(\omega) = I(\omega) \times A = \frac{e(\omega) \times e(\omega)^*}{2\eta_0} A$$

And we can do an inverse Fourier transform to find the time-domain function of the electric field.

$$e(t) = \sqrt{\frac{2\eta_0 P_0}{A}} e^{j\omega_0 t} \sum_{n=-\infty}^{+\infty} \mathrm{sech}(\frac{n\omega_c}{\Delta\omega}) e^{jn\omega_c t}$$

Now I have to evaluate that big summation. As shown in p.301, we can approximate this summation with integral.

Let

$$n\omega_c = x$$

$$dx/_{\omega_c} = 1$$

$$\sum_{n=-\infty}^{+\infty} \mathrm{sech}(\frac{n\omega_c}{\Delta\omega}) e^{jn\omega_c t} \sim \frac{1}{\omega_c}\int_{-\infty}^{\infty} \mathrm{sech}(\frac{x}{\Delta\omega}) e^{jxt}\, dx$$

Since the sech function is an even function, we can further simplify the integral.

$$\frac{1}{\omega_c}\int_{-\infty}^{\infty} \mathrm{sech}(\frac{x}{\Delta\omega}) e^{jxt} dx = \frac{1}{\omega_c}\int_{-\infty}^{\infty} \mathrm{sech}\left(\frac{x}{\Delta\omega}\right)\left(\cos(xt) + jsin(xt)\right)dx$$

$$= \frac{2}{\omega_c}\int_{0}^{\infty} \mathrm{sech}\left(\frac{x}{\Delta\omega}\right)\cos(xt) dx$$

Solve this integral by Mathematica.

$$\frac{2}{\omega_c}\int_{-\infty}^{\infty} \mathrm{sech}(\frac{x}{\Delta\omega}) e^{jxt} dx = \frac{2}{\omega_c} \times \frac{1}{2}\pi\Delta\omega \times \mathrm{sech}\left(\frac{\pi\Delta\omega t}{2}\right) = \frac{\pi\Delta\omega}{\omega_c} \mathrm{sech}\left(\frac{\pi\Delta\omega t}{2}\right)$$

$$e(t) = \sqrt{\frac{2\eta_0 P_0}{A}} e^{j\omega_0 t} \frac{\pi\Delta\omega}{\omega_c} \mathrm{sech}\left(\frac{\pi\Delta\omega t}{2}\right)$$

$$p(t) = P_0 \left(\frac{\pi\Delta\omega}{\omega_c}\right)^2 sech^2(\frac{\pi\Delta\omega t}{2})$$

Finally, we can calculate the average power.

$$Average\ Power = \frac{Power\ within\ one\ period}{period}$$

The period can be easily calculated through rough trip time.

$$\tau_{RT} = \frac{2\pi}{\omega_c}$$

$$\langle P \rangle = \frac{\int_{-\infty}^{\infty} p(t)dt}{\tau_{RT}} = 200mW$$

(b)

From the above equation, we have the peak power when sech equals to 1.

$$P_{peak} = P_0 \left(\frac{\pi\Delta\omega}{\omega_c}\right)^2 = 9869.6W$$

(c)

It’s obvious that the lineshape is determined by the sech function. So FWHM is just the sech function equals to half.

$$sech^2\left(\frac{\pi\Delta\omega t}{2}\right) = \frac{1}{2}$$

$$\operatorname{sech}\left(\frac{\pi\Delta\omega t}{2}\right) = \frac{1}{\sqrt{2}}$$

Solve it by numerical method. We get

$$t_{\frac{1}{2}} = 8.93 \times 10^{-11}s$$

$$FWHM = 2t_{\frac{1}{2}} = 0.1786nsec$$

Problem 5.8 (Verdeyen Problem # 9.31)

(a)

This problem is similar to Verdeyen Problem # 9.6c.

$$\frac{df}{dz} = \{\frac{\gamma_0}{1+f} - \alpha\}f$$

$$\frac{df}{dz} = \frac{\gamma_0 - \alpha(1+f)}{1+f}f$$

$$\frac{1+f}{f(\gamma_0 - \alpha(1+f))}df = dz$$

$$\frac{1+f}{\gamma_0 - \alpha(1+f)} = \frac{c}{f} + \frac{d}{(\gamma_0 - \alpha) - \alpha f}$$

$$c = \frac{1}{\gamma_0 - \alpha}$$

$$d = \frac{\gamma_0}{(\gamma_0 - \alpha)}$$

$$\left[\frac{1}{f} + \frac{\gamma_0}{(\gamma_0 - \alpha) - \alpha f}\right]df = (\gamma_0 - \alpha)dz$$

$$\int_{f_1}^{f_2}\left[\frac{1}{f} + \frac{\gamma_0}{(\gamma_0 - \alpha) - \alpha f}\right]df = (\gamma_0 - \alpha)l_g$$

$$\ln\left(\frac{f_2}{f_1}\right) - \frac{\gamma_0}{\alpha}\ln\left(\frac{(\gamma_0 - \alpha) - \alpha f_2}{(\gamma_0 - \alpha) - \alpha f_1}\right) = (\gamma_0 - \alpha)l_g$$

So the problem is asking for extractable power, meaning the additional power we get from the amplifier. Let's assume $f_2 = f_1 + f_e$.

The above equation should be rewritten.

$$\ln\left(1 + \frac{f_e}{f_1}\right) - \frac{\gamma_0}{\alpha}\ln\left(\frac{(\gamma_0 - \alpha) - \alpha f_2}{(\gamma_0 - \alpha) - \alpha f_1}\right) = (\gamma_0 - \alpha)l_g$$

And we use an approximation for $\ln(1+x) \sim x$. Since we're considering the maximum extractable power, it's reasonable to assume the f_1 is a large number, meaning the approximation is valid.

$$\frac{f_e}{f_1} - \frac{\gamma_0}{\alpha}\ln\left(1 - \frac{\alpha}{\gamma_0 - \alpha}f_2\right) + \frac{\gamma_0}{\alpha}\ln\left(1 - \frac{\alpha}{\gamma_0 - \alpha}f_1\right) = (\gamma_0 - \alpha)l_g$$

$$\frac{f_e}{f_1} + \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}f_2 - \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}f_1 = \frac{f_e}{f_1} + \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}(f_1 + f_e) - \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}f_1$$

$$= \frac{f_e}{f_1} + \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}f_e = (\gamma_0 - \alpha)l_g$$

$$\frac{f_e}{f_1} + \frac{\gamma_0}{\alpha}\frac{\alpha}{\gamma_0 - \alpha}f_e \sim \frac{\gamma_0}{\gamma_0 - \alpha}f_e = (\gamma_0 - \alpha)l_g$$

$$f_e = \frac{(\gamma_0 - \alpha)^2}{\gamma_0} l_g$$

$$I_{e,max} = \frac{(\gamma_0 - \alpha)^2}{\gamma_0} l_g \times I_s$$

(b)

Now we have to apply it into a ring laser.

First, we defined a few parameters according to the problem.

$$g_{th} = (\gamma_{th} - \alpha) l_g$$

$$g_0 = (\gamma_0 - \alpha) l_g$$

$$\delta = \frac{\alpha}{\gamma_0 - \alpha}$$

From (9.2.4), input and output intensities can be related.

$$f_1 = S f_2$$

$$S = surviving\ factor$$

In order to reach threshold condition, we have one more relation.

$$S \times e^{g_{th}} = 1$$

$$g_{th} = \ln\left(\frac{1}{S}\right) = -\ln(S)$$

$$f_1 = S f_2 = e^{-g_{th}} f_2$$

Plug in all the parameters into following equation we solved in the previous problem.

$$\ln\left(\frac{f_2}{f_1}\right) - \frac{\gamma_0}{\alpha} \ln\left(\frac{(\gamma_0 - \alpha) - \alpha f_2}{(\gamma_0 - \alpha) - \alpha f_1}\right) = (\gamma_0 - \alpha) l_g$$

$$\ln\left(\frac{f_2}{f_1}\right) + \frac{\delta + 1}{\delta} \ln\left(\frac{1 - \delta f_1}{1 - \delta f_2}\right) = (\gamma_0 - \alpha) l_g$$

$$g_{th} + \frac{\delta + 1}{\delta} \ln\left(\frac{1 - \delta e^{-g_{th}} f_2}{1 - \delta f_2}\right) = g_0$$

$$\ln\left(\frac{1 - \delta e^{-g_{th}} f_2}{1 - \delta f_2}\right) = \frac{\delta}{\delta + 1} (g_0 - g_{th})$$

$$\frac{1 - \delta e^{-g_{th}} f_2}{1 - \delta f_2} = e^{\frac{\delta}{\delta+1}(g_0 - g_{th})}$$

$$f_2 = \frac{1 - e^{\frac{\delta}{\delta+1}(g_0 - g_{th})}}{\delta\left(e^{-g_{th}} - e^{\frac{\delta}{\delta+1}(g_0 - g_{th})}\right)}$$

Output can be calculated from (9.2.6).

$$I_{out} = T_b T_2 I_s f_2 = T_b T_2 I_s \frac{1 - e^{\frac{\delta}{\delta+1}(g_0 - g_{th})}}{\delta\left(e^{-g_{th}} - e^{\frac{\delta}{\delta+1}(g_0 - g_{th})}\right)}$$

Now if $\delta \ll 1$, it means the factor power of exp is very small. We can make the following approximation.

$$e^{x} \approx 1 + x \; if \; x \ll 1$$

$$I_{out} = T_b T_2 I_s \frac{1 - 1 - \frac{\delta}{\delta + 1}(g_0 - g_{th})}{\delta(e^{-g_{th}} - 1 - \frac{\delta}{\delta + 1}(g_0 - g_{th}))} \approx T_b T_2 I_s \frac{1}{\delta + 1} \frac{(g_0 - g_{th})}{(1 - e^{-g_{th}})}$$

$$= T_b T_2 I_s \frac{\gamma_0 - \alpha}{\gamma_0} \frac{(g_0 - g_{th})}{(1 - e^{-g_{th}})}$$

Problem 5.9 (Verdeyen Problem # 9.36)

(a)

$$\frac{c}{2d} = 200MHz$$

$$d = \frac{c}{2 \times 200MHz} = 0.75m$$

(b)

$$T = \frac{2d}{c} = 5nsec$$

(c)

$$I(t) = \frac{E(t) \times E(t)^*}{2\eta_0}$$

Let's first try to find the analytical form of electric field.

$$E(t) = E_0 e^{j\omega_0 t} \sum_{-\infty}^{\infty} \frac{\sin[\frac{n\omega_c}{\Delta\omega}] e^{jn\omega_c t}}{\frac{n\omega_c}{\Delta\omega}}$$

Now we're using the same trick to replace the summation with integral.

$$\sum_{-\infty}^{\infty} \frac{\sin[\frac{n\omega_c}{\Delta\omega}] e^{jn\omega_c t}}{\frac{n\omega_c}{\Delta\omega}} = \int_{-\infty}^{\infty} \frac{\sin[\frac{x}{\Delta\omega}]}{\frac{x}{\Delta\omega}} e^{jxt} \frac{dx}{\omega_c}$$

By observation, the original time domain should be a square pulse function, because the Fourier transform of square pulse function is sinc function.

$$\omega_c{}^{-1} \int_{-\infty}^{\infty} \frac{\sin[\frac{x}{\Delta\omega}]}{\frac{x}{\Delta\omega}} e^{jxt} dx = 2\pi\omega_c{}^{-1} \frac{\Delta\omega}{2} [\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{\Delta\omega} \frac{\sin\left[\frac{x}{\Delta\omega}\right]}{\frac{x}{\Delta\omega}} e^{jxt} dx]$$

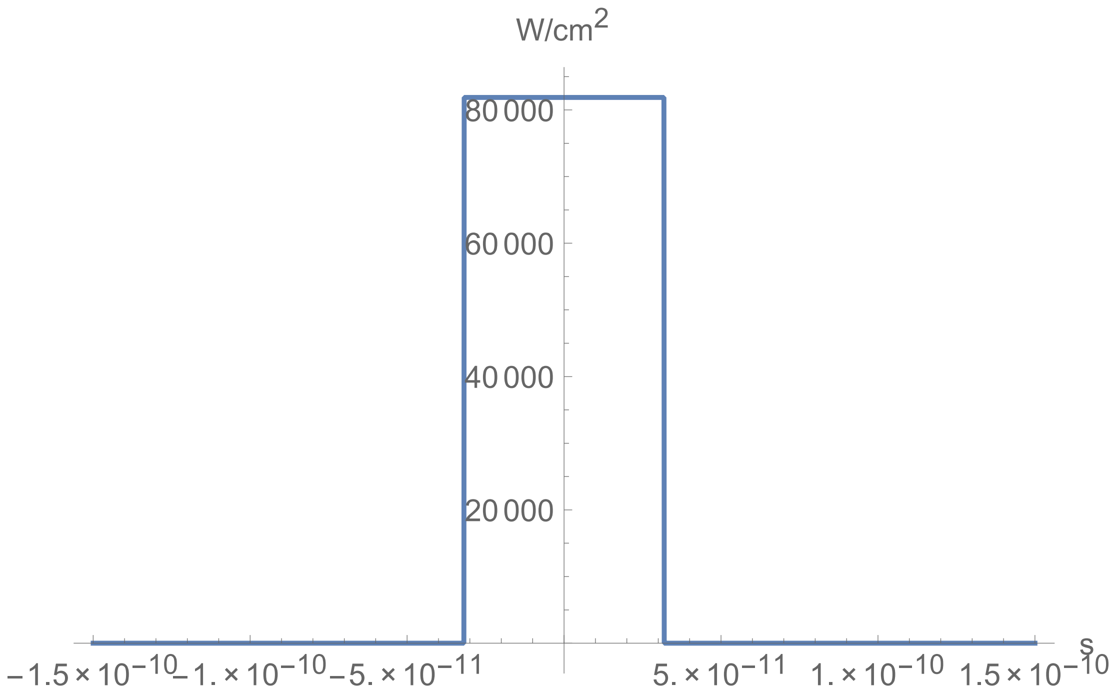
$$= 2\pi\omega_c{}^{-1} \frac{\Delta\omega}{2} \times rect(\frac{\Delta\omega}{2} t)$$

$$rect\left(\frac{\Delta\omega}{2} t\right) = 1 \; for |t| < \frac{1}{\Delta\omega}$$

$$rect\left(\frac{\Delta\omega}{2} t\right) = 0 \; for |t| > \frac{1}{\Delta\omega}$$

$$I(t) = \frac{E_0{}^2}{2\eta_0} \pi^2 (\frac{\Delta\omega}{\omega_c})^2 (rect(\frac{\Delta\omega}{2} t))^2$$

Now plot it with Mathematica.



This pulse should repeat for a period of 5nsec.

$$I_{max} = 81868W/cm^2$$

(c)

This is a simple question. The output pulse energy should be the product of the transmission function and the pulse itself.

$$Output\ energy\ 1 = I_{max}\int_{\frac{-1}{\Delta\omega}}^{\frac{1}{\Delta\omega}} cos^2\left(\frac{\pi t}{T}\right)dt$$

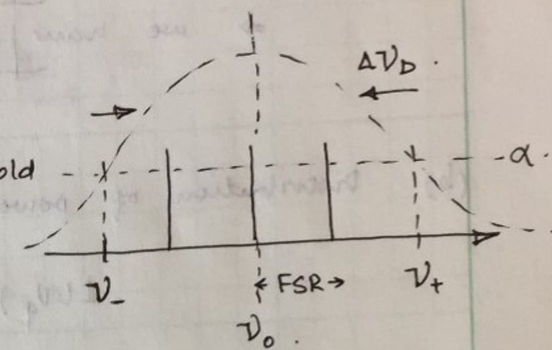
If there is no shutter, meaning $Output\ power\ 0 = I_{max}\frac{2}{\Delta\omega}$

$$Loss = \frac{Output\ power\ 0 - Output\ energy\ 1}{Output\ power\ 0} = 0.000133$$

5.10 Doppler broadened $\rightarrow$ $\gamma_0(\nu) = \gamma_0(\nu_0) \exp\left\{-4\ln 2 \left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2\right\}$.

given : $\Delta\nu = 5 \times 10^9$ Hz.

$d = 100$ cm



(a) FSR of cavity modes

$= \frac{c}{2d}$

$= \frac{3 \times 10^{10}}{200} = 1.5 \times 10^8$ Hz.

No. of modes $= \frac{\text{Gain width at threshold.}}{\text{FSR.}}$

Given $\rightarrow$ peak = 3 times threshold.

(or) $\gamma_0(\nu_0) = 3\alpha. = 3\gamma_0(\nu_{+/-})$

we want ν_+/ν_- to get width.

$\gamma_0(\nu_\pm) = \gamma_0(\nu_0) \cdot \exp\left\{-4\ln 2 \left(\frac{\nu_{+/-}-\nu_0}{\Delta\nu_D}\right)^2\right\}$.

$\exp\left\{-4\ln 2 \left(\frac{\nu_\pm-\nu_0}{\Delta\nu_D}\right)^2\right\} = \frac{1}{3}$.

$-4\ln 2 . \left(\frac{\nu_\pm-\nu_0}{\Delta\nu_0}\right)^2 = \ln 1/3$.

$\frac{\nu_\pm-\nu_0}{\Delta\nu_0} = \sqrt{0.3962} = 0.629$.

$\nu_\pm - \nu_0 = 0.629 \times \Delta\nu_0$.

$\therefore \nu_+ - \nu_- = 2 \times 0.629 \times \Delta\nu_0$

$= 6.29 \times$ GHZ

$\therefore$ No. of modes $= \dfrac{6.29 \text{ GHZ}}{150 \text{ MHZ}}$

$= 41.9.$

$\Rightarrow$ we have $\boxed{41 \text{ modes}}$ above the threshold.

(b) Distribution of power among these modes :

$$I(\nu_q) = \underbrace{I(\nu_0)}_{I_0} \exp\left\{-4 \ln 2 \left(\frac{\nu_q - \nu_0}{\underbrace{\Delta\nu_0}_{\text{FWHM}}}\right)^2\right\} \quad : \quad e^{-2r^2/w_0^2}$$

Intensity is also gaussian in beam spot.

Since inhomo. broadening.

$$\underbrace{\frac{I_\nu}{I_s}}_{\text{at thresh}} \sim \left(\frac{r_0(\nu)}{\alpha}\right)^2 - 1.$$

$$\frac{I_0 \cancel{3^2} \exp\left\{-4 \ln 2 \left(\frac{\nu_q - \nu_0}{\Delta\nu_0}\right)^2\right\}}{I_s \exp\left\{-4 \ln 2 \left(\frac{\nu_q - \nu_0}{\Delta\nu_0}\right)^2\right\}} \approx = (3)^2 - 1.$$

$$3^2 \exp\left\{-8 \ln 2 \left(\frac{\nu_q - \nu_0}{\Delta\nu_0}\right)^2\right\} = \frac{3^2}{2} - \frac{1}{2}$$

$$\Rightarrow \Delta\nu_0 = \Delta\nu_D \left\{\frac{\ln 18/10}{2 \ln 2}\right\} = 0.651\,\Delta\nu_D.$$

Peak power $\rightarrow$ integrate intensity over the total area.

$$\Rightarrow \quad P_0 = I_0 \int_0^{\infty} e^{-2r^2/w_0^2}\, r\,dr = I_0 \cdot \frac{\pi w_0^2}{2}.$$

We are given average power!

$$\frac{\langle I \rangle}{I_0} = \frac{1}{2}\left(\frac{\pi}{\ln 2}\right)^{1/2}\left(\frac{\Delta\nu_{osc}}{\Delta\nu_D}\right) = \frac{0.651}{2}\left(\frac{\pi}{\ln 2}\right)^{1/2}$$

$$\Rightarrow \quad P_0 = 2\langle P \rangle \cdot \frac{2}{0.651}\left(\frac{\ln 2}{\pi}\right)^{1/2}$$

$$= 2 \times 4\text{ W} \times \frac{2}{0.651} \times \left(\frac{\ln 2}{\pi}\right)^{1/2}$$

$$\boxed{P_0 = 0.141\text{ W}} \Rightarrow \text{Peak power}$$

10/10

$$P_{peak} = \left(\frac{\pi}{\ln 2}\right)^{1/2} 0.651 \times 4\text{ W}$$

$$\boxed{P_{peak} = 354\text{ W}}$$

(d) Pulse width can be given by : $\Delta t\, \Delta\omega = 4 \ln 2$.

$$\Rightarrow \quad \Delta t = \frac{4 \ln 2}{\Delta \omega}$$

$$= \frac{4 \ln 2}{2\pi \times 0.651 \times 5 \times 10^9}$$

$$\boxed{= 1.35 \times 10^{-13}\text{ s}}$$

(e) Photon density $= \dfrac{\text{Power}}{h\nu} = \cancel{I_0 \int_0^{\infty} e}\; I_0 \cdot \dfrac{\pi \cdot (2\text{mm})^2}{2} \times \dfrac{1}{h\nu}$.